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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* Newton

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. Euler

Ceux qui aiment l'Analyse, verront avec plaisir la Méchanique en devenir une nouvelle branche ... Lagrange

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On Steady-State Solutions of the Navier-Stokes Partial Differential Equations

ROBERT FINN

1. Definitions and Summary

A three dimensional vector field $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(x_1, x_2, x_3)$ defined in a region $\mathscr{E}$ of Euclidean space will be said to represent in $\mathscr{E}$ the *velocity field of a time independent flow of a viscous incompressible fluid,* provided, a) all derivatives of $\boldsymbol{u}(\boldsymbol{x})$ up to the second order exist and are continuous in $\mathscr{E}$, and b) there exists a scalar $p(\boldsymbol{x})$ with continuous first partial derivatives, such that the Navier-Stokes partial differential equations,

$$\begin{aligned} \Delta \boldsymbol{u} - \nabla p &= (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u}, \\ \operatorname{div} \boldsymbol{u} &= 0, \end{aligned} \tag{1}$$

are satisfied by the functions $\boldsymbol{u}(\boldsymbol{x})$, $p(\boldsymbol{x})$ in $\mathscr{E}$. The notation conforms with the classical usage of vector analysis. Certain constants which appear in the usual formulation of the equations (1) have been arbitrarily set equal to unity, because they play no role in the context of this paper.

Such a vector field $\boldsymbol{u}(\boldsymbol{x})$ will be said to define a *solution of an exterior boundary value problem* provided a) and b) are satisfied, and provided c) $\mathscr{E}$ is the exterior of a bounded region G whose boundary surface Σ consists of a finite number of components $\Sigma_1, \ldots, \Sigma_m$, d) on each component Σ_i, $\boldsymbol{u}(\boldsymbol{x})$ assumes prescribed values $\boldsymbol{u}_i^*$, and e) $\boldsymbol{u}(\boldsymbol{x})$ tends to a prescribed constant vector $\boldsymbol{u}_0$ as $\boldsymbol{x} \to \infty$. It will be assumed throughout that the surfaces Σ_i and the prescribed data $\boldsymbol{u}_i^*$ are smooth. For precise assumptions, sufficient in the present context, see ODQVIST [*1*]. We shall also make the (physically natural) assumption on the boundary data $\boldsymbol{u}_i^*$, that the surface integral

$$\int_{\Sigma_i} \boldsymbol{u}_i^* \cdot d\boldsymbol{S} = 0 \tag{2}$$

for each component Σ_i. We note in particular that (2) is satisfied when $\boldsymbol{u}_i^* = 0$, $i = 1, \ldots, m$. A solution $\boldsymbol{u}(\boldsymbol{x})$ corresponding to such data admits a physical interpretation as a flow of a viscous fluid past a rigid obstacle.

In a remarkable and penetrating study of the Navier-Stokes equations, LERAY [*2*] has proved that for any region $\mathscr{E}$ of the type considered, there exists a solution of the exterior boundary value problem in a generalized sense. Precisely,

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This work was prepared while the author was a Guggenheim Fellow, 1958–1959. The author is indebted to the Guggenheim Memorial Foundation, and also to the California Institute of Technology, for support and encouragement offered in many ways.

LERAY has demonstrated the existence in $\mathcal{E}$ of a vector field $\boldsymbol{u}(\boldsymbol{x})$ satisfying a), b), c), d), and such that $\boldsymbol{u}(\boldsymbol{x})$ is, in every compact subregion of $\mathcal{E}$, the uniform limit, as $N \to \infty$ along a suitable sequence of values, of solutions $\boldsymbol{u}_N(\boldsymbol{x})$ of the following interior boundary value problem: $\boldsymbol{u}_N(\boldsymbol{x}) = \boldsymbol{u}_i^*(\boldsymbol{x})$ on Σ_i, $\boldsymbol{u}_N(\boldsymbol{x}) = \boldsymbol{u}_0$ on the boundary Σ_N of a sphere S_N of radius N and center at the origin of coordinates. Further, the Dirichlet integrals of $\boldsymbol{u}_N(\boldsymbol{x})$ over the regions $\mathcal{E}_N = S_N \cap \mathcal{E}$ are uniformly bounded,

$$D[\boldsymbol{u}_N] = \int_{\mathcal{E}_N} |\nabla \boldsymbol{u}_N|^2 dV < K \tag{3}$$

where K is independent of N.

LERAY did not prove that his solution $\boldsymbol{u}(\boldsymbol{x})$ tends to $\boldsymbol{u}_0$ as a limit except in the special case $\boldsymbol{u}_0 = 0$, a case which seems to be without physical interest. The chief purpose of the present paper is to fill this gap. We shall prove the existence of a function $R(\varepsilon)$ such that for any $\varepsilon > 0$, the limit function $\boldsymbol{u}(\boldsymbol{x})$ has the property $|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_0| < \varepsilon$ whenever $|\boldsymbol{x}| > R(\varepsilon)$. We shall also obtain an integral representation which is valid for any solution of an exterior boundary value problem obtained by the method of LERAY[1]. Our procedure exploits a reformulation of the equations (1) suggested by a linearization due to OSEEN [*3*], and a form of Green's identity used by ODQVIST [*1*] in his study of the linearization of STOKES. We are led to an integral representation for the solution $\boldsymbol{u}_N$ which differs slightly from the one employed by LERAY and which proves to be advantageous for obtaining the necessary *a priori* estimates on the approximating solutions. We shall prove, then, the following result.

Theorem 1. *Corresponding to any region $\mathcal{E}$ of the type considered, and to any prescribed data $\boldsymbol{u}_0$ and $\boldsymbol{u}_i^*$ satisfying* (2), *there exists at least one solution of the exterior boundary value problem* a), b), c), d), e). *The pressure $p(\boldsymbol{x})$ corresponding to this solution tends to a finite limit p_0 as $\boldsymbol{x} \to \infty$. The solution and pressure admit the representations*

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0 &- \int_{\mathcal{E}} \{\mathbf{X}(\boldsymbol{x},\boldsymbol{y}) \cdot [\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}_0] \cdot \nabla \boldsymbol{u}(\boldsymbol{y})\} dV_{\boldsymbol{y}} + \\ &+ \int_{\Sigma} \{\mathbf{X}(\boldsymbol{x},\boldsymbol{y}) \cdot T\boldsymbol{u}(\boldsymbol{y}) - [\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}_0] \cdot T\mathbf{X}(\boldsymbol{x},\boldsymbol{y})\} dS_{\boldsymbol{y}} - \\ &- \int_{\Sigma} [\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}_0] \cdot \mathbf{X}(\boldsymbol{x},\boldsymbol{y}) (\boldsymbol{n}_0 \cdot \boldsymbol{n}) dS_{\boldsymbol{y}}, \end{aligned} \tag{4}$$

$$\begin{aligned} p(\boldsymbol{x}) = p_0 &- \int_{\mathcal{E}} \{\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y}) \cdot [\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}_0] \cdot \nabla \boldsymbol{u}(\boldsymbol{y})\} dV_{\boldsymbol{y}} + \\ &+ \int_{\Sigma} \{\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y}) \cdot T\boldsymbol{u}(\boldsymbol{y}) - [\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}_0] \cdot T\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y})\} dS_{\boldsymbol{y}} - \\ &- \int_{\Sigma} [\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}_0] \cdot \boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y}) (\boldsymbol{n}_0 \cdot \boldsymbol{n}) dS_{\boldsymbol{y}}. \end{aligned} \tag{4'}$$

Here $\boldsymbol{n}_0$ denotes a unit vector in the direction of $\boldsymbol{u}_0$. The tensor $\mathbf{X}$ and vector $\boldsymbol{\psi}$ are explicitly known and will be defined in the following sections, as will the operator T.

As a corollary of our method, we are able to obtain a characterization at infinity of a class of solutions of (1) which includes the solutions of LERAY.

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[1] The representation is valid for a class of solutions which may be more general, *cf.* Theorem 2.

Theorem 2. *Let* $\boldsymbol{u}(\boldsymbol{x})$ *denote a solution of* (1) *in a neighborhood* $\mathscr{E}$ *of infinity, and suppose that the Dirichlet integral of* $\boldsymbol{u}(\boldsymbol{x})$ *in* $\mathscr{E}$ *is finite,* $\int\limits_{\mathscr{E}} |\nabla \boldsymbol{u}|^2 dV < \infty$. *Then there exists a constant vector* $\boldsymbol{u}_0$ *such that* $\boldsymbol{u}(\boldsymbol{x}) \to \boldsymbol{u}_0$ *as* $\boldsymbol{x} \to \infty$, *and* $\boldsymbol{u}(x)$ *admits, for any surface* Σ *enclosing the complement of* $\mathscr{E}$, *the representation* (4), (4').

The critical new step in the proof of Theorem 2 is the application of a lemma of Payne & Weinberger [*4*], which provides an estimate for integral means at infinity of functions which possess bounded Dirichlet integrals.

One might expect that the representation (4) could be used to obtain precise asymptotic estimates on the behavior of the solution at infinity. The author is as yet unable to do this, the chief technical difficulty being that the components of the fundamental solution tensor (9) are not square integrable over the entire space. Our best result is the following:

Theorem 3. *Let* $\boldsymbol{u}(\boldsymbol{x})$ *denote a solution of* (1) *exterior to* Σ *and having finite Dirichlet integral,* $\int |\nabla \boldsymbol{u}|^2 dV < \infty$. *Let* $\boldsymbol{u}_0$ *be the constant vector whose existence is assured by Theorem 2, and suppose that* $|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_0| < C|\boldsymbol{x}|^{-\alpha}$, $\alpha > \frac{1}{2}$, *as* $\boldsymbol{x} \to \infty$, *for some constant* C. *Then there is a constant* $\overline{C}$ *such that* $|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_0| < \overline{C}|\boldsymbol{x}|^{-1}$ *as* $\boldsymbol{x} \to \infty$.

The same method yields also an improved estimate for $|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_0|$ outside a limited "wake" region, but we shall not pursue this matter here.

From a conceptual point of view, it would probably be best to prove first Theorem 2. Theorem 1 follows easily from this result and from the work of Leray. However, the proof of Theorem 2 requires ideas which are unnecessary for the existence theorem, and we shall therefore provide first, in Sections 2, 3, and 4, the proof of Theorem 1. We prove Theorem 2 in Section 5, making reference to the developments of the earlier sections. Section 6 is devoted to a proof of the final theorem, which follows easily from the integral representation (4).

Throughout this paper the symbols C and K will be used to denote positive constants, the values of which may change even within a given context. Thus, from the inequality $\psi \leqq C(R^2+1)^{\frac{1}{2}}$ we may conclude $\psi \leqq CR$ for $R \geqq 1$. Similarly, the symbol $\varepsilon(R)$ will denote a quantity which tends to zero as $R \to \infty$.

Finally we remark that the methods of this paper do not seem to suffice to prove the corresponding results for two dimensional solutions of (1). In view of the known singular behavior at infinity of the solutions of the Stokes linearization of (1) (*cf.* [*5*]), this does not seem surprising. Although this singularity does not appear in solutions of the Oseen linearized equations, there may be cases in which the non-linearity in (1) prevents the solution from tending to its limit quickly enough to permit the Oseen linearization to be a valid approximation at infinity.

2. The Integral Representation

Consider a divergence-free vector field $\boldsymbol{u}(\boldsymbol{x}) = (u_1, u_2, u_3)$ and a corresponding scalar $p(\boldsymbol{x})$. We define the second order tensor $T\boldsymbol{u}$ by the relation

$$(T\boldsymbol{u})_{ij} = -p\,\delta_{ij} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{5}$$

Let us also define
$$L\boldsymbol{u} = \Delta \boldsymbol{u} - \nabla p.$$

We then find easily the identities, valid for any divergence-free fields $\boldsymbol{u}$ and $\boldsymbol{v}$ defined in a region V with sufficiently smooth boundary S,

$$(6)\qquad \begin{aligned} &\text{a)}\quad \int_V \boldsymbol{v}\cdot L\boldsymbol{u}\,dV = -\tfrac{1}{2}\int_V (\operatorname{def}\boldsymbol{v}\cdot\operatorname{def}\boldsymbol{u})\,dV + \int_S \boldsymbol{v}\cdot T\boldsymbol{u}\,dS,\\ &\text{b)}\quad \int_V (\boldsymbol{v}\cdot L\boldsymbol{u} - \boldsymbol{u}\cdot L\boldsymbol{v})\,dV = \int_S (\boldsymbol{v}\cdot T\boldsymbol{u} - \boldsymbol{u}\cdot T\boldsymbol{v})\,dS.\end{aligned}$$

These identities are to be understood in the sense that

$$(\operatorname{def}\boldsymbol{v}\cdot\operatorname{def}\boldsymbol{u}) = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \qquad \boldsymbol{v}\cdot T\boldsymbol{u} = v_i (T\boldsymbol{u})_{ij}\, n_j,$$

$\boldsymbol{n} = (n_1, n_2, n_3)$ = unit exterior directed normal on S, with summation extended over repeated indices. Let now $\mathbf{X}(\boldsymbol{x},\boldsymbol{y}) = (\mathsf{X}_{ij})$, $\mathbf{P}(\boldsymbol{x},\boldsymbol{y}) = (\mathsf{P}_i)$ denote a 3×3 matrix and 3-vector, respectively, which become singular at $\boldsymbol{x}=\boldsymbol{y}$ in such a way that

$$\delta_{ik} = \lim_{r\to 0}\int_{\Sigma_r}\left\{\mathsf{P}_k\,\delta_{ij} - \left(\frac{\partial \mathsf{X}_{ik}}{\partial x_j} + \frac{\partial \mathsf{X}_{jk}}{\partial x_i}\right)\right\} n_j\, dS_{\boldsymbol{y}}$$

where Σ_r denotes the surface of a sphere of radius r about $\boldsymbol{x}$ as center. If, in addition, the row vectors of $\mathbf{X}$ are divergence free, we obtain from (6b) the representation

$$(7)\qquad \boldsymbol{u}(\boldsymbol{x}) = -\int_V (\mathbf{X}\cdot L\boldsymbol{u} - \boldsymbol{u}\cdot L\mathbf{X})\,dV_{\boldsymbol{y}} - \int_S (\boldsymbol{u}\cdot T\mathbf{X} - \mathbf{X}\cdot T\boldsymbol{u})\,dS_{\boldsymbol{y}}$$

valid at interior points of V for any divergence-free vector field $\boldsymbol{u}(\boldsymbol{x})$ in V. It will be convenient for us to choose as quantities $(\mathbf{X},\mathbf{P})$ a *fundamental solution tensor* for the adjoint system corresponding to the differential equations of OSEEN,

$$(8)\qquad \begin{aligned} \Delta\boldsymbol{u} - \boldsymbol{u}_0\cdot\nabla\boldsymbol{u} - \nabla p &= 0,\\ \operatorname{div}\boldsymbol{u} &= 0.\end{aligned}$$

Such a tensor has been given explicitly by OSEEN. It is obtained from the relations (*cf.* [3], p. 34),

$$(9)\qquad \begin{aligned} \mathsf{X}_{ij} &= \delta_{ij}\,\Delta\Phi - \frac{\partial^2\Phi}{\partial x_i\,\partial x_j},\\ \mathsf{P}_j &= -\frac{\partial}{\partial x_j}\left[\Delta\Phi - \boldsymbol{u}_0\cdot(\nabla\Phi)\right],\\ \Phi &= \frac{1}{\sigma}\int_0^{\sigma s}\frac{1-e^{-\alpha}}{\alpha}\,d\alpha,\end{aligned}$$

$$\sigma = \frac{|\boldsymbol{u}_0|}{2}, \qquad s = r + \frac{\boldsymbol{u}_0\cdot(\boldsymbol{y}-\boldsymbol{x})}{|\boldsymbol{u}_0|}, \qquad r = |\boldsymbol{y}-\boldsymbol{x}|.$$

We have here neglected an inessential constant factor.

This tensor has the property, in addition to those mentioned above, that for every fixed j, the vector X_{ij} and corresponding scalar P_{ji} considered as functions of $\boldsymbol{y}$, define a solution of the adjoint system

$$(8')\qquad \begin{aligned} \Delta\boldsymbol{u} + \boldsymbol{u}_0\cdot\nabla\boldsymbol{u} - \nabla p &= 0,\\ \operatorname{div}\boldsymbol{u} &= 0\end{aligned}$$

whenever $\boldsymbol{x}\neq\boldsymbol{y}$.

We now rewrite the equations (1) in the form

$$\Delta(\boldsymbol{u}-\boldsymbol{u}_0)-\boldsymbol{u}_0\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}_0)-\nabla p=(\boldsymbol{u}-\boldsymbol{u}_0)\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}_0),$$
$$\operatorname{div}(\boldsymbol{u}-\boldsymbol{u}_0)=0.$$

Throughout this paper we shall fix attention on the vector $\boldsymbol{u}-\boldsymbol{u}_0$, which, changing notation, we shall write simply as $\boldsymbol{u}$. Further, we make the normalization[2], inessential to the mathematical development, that $\boldsymbol{u}_0=(1,0,0)$. The equations (1) transform to

$$\Delta\boldsymbol{u}-\frac{\partial\boldsymbol{u}}{\partial x_1}-\nabla p=\boldsymbol{u}\cdot\nabla\boldsymbol{u},\qquad\qquad(1')$$
$$\operatorname{div}\boldsymbol{u}=0,$$

and we find the integral representation, valid for every solution of (1'),

$$(10)\quad \boldsymbol{u}(\boldsymbol{x})=-\int\limits_V\left(u\cdot\frac{\partial\mathbf{X}}{\partial x_1}+\mathbf{X}\cdot\frac{\partial\boldsymbol{u}}{\partial x_1}\right)dV-\int\limits_V\mathbf{X}\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\,dV+\int\limits_S(\mathbf{X}\cdot T\boldsymbol{u}-\boldsymbol{u}\cdot T\mathbf{X})\,dS.$$

We choose for S the annular region $\mathscr{E}_N$ bounded by the given boundary Σ and by a sphere Σ_N of radius N sufficiently large that Σ lies within Σ_N. We consider in $\mathscr{E}_N$ a solution $\boldsymbol{u}_N(\boldsymbol{x})$ of (1') such that $\boldsymbol{u}_N(\boldsymbol{x})$ assumes the prescribed values $\boldsymbol{u}^*$ on Σ, and such that $\boldsymbol{u}_N(\boldsymbol{x})=0$ on Σ_N. From the work of LERAY [*2*], we are assured of the existence of $\boldsymbol{u}_N(\boldsymbol{x})$, and we know that the Dirichlet integral $D_N[\boldsymbol{u}_N]<K$ for all N. Since the integrand in the first term on the right of (10) is a divergence, we find

$$(11)\quad\begin{aligned}\boldsymbol{u}_N(x)&=-\int\limits_{\mathscr{E}_N}\mathbf{X}\cdot(\boldsymbol{u}_N\cdot\nabla\boldsymbol{u}_N)\,dV+\\&\quad+\int\limits_{\Sigma}(\mathbf{X}\cdot T\boldsymbol{u}_N-\boldsymbol{u}_N\cdot T\mathbf{X}-\boldsymbol{u}_N\cdot\mathbf{X}n_1)\,dS+\int\limits_{\Sigma_N}\mathbf{X}\cdot T\boldsymbol{u}_N\,dS,\end{aligned}$$

where the symbol n_1 denotes the x_1 component of a unit normal to Σ.

We note, for later reference, some properties of $\mathbf{X}$ and of $\mathbf{P}$. Recalling the normalization $\boldsymbol{u}_0=(1,0,0)$, we find from (9),

(12)
- i) $|\mathsf{X}_{ij}|<\frac{C}{r}$ in a neighborhood of $r=0$, and also in the paraboloidal "wake" region defined by $r+(y_1-x_1)\leqq\text{const.}$
- ii) $|\mathsf{X}_{ij}|<\frac{C}{r^2}$ for $r+(y_1-x_1)\geqq C\log r,\ r\to\infty$.
- iii) for all derivatives of order n of the components of $\mathbf{X}$, we have always $|\mathsf{X}_{ij}^{(n)}|<\frac{C}{r^{1+n/2}},\ r\to\infty$.
- iv) $|\mathsf{P}_j|<\frac{C}{r^2},\ r\to 0$ and $r\to\infty$.
- v) For a sphere of radius r about $\boldsymbol{x}$, we have $\int\limits_{\Sigma_r}\mathsf{X}_{ij}^2(\boldsymbol{x},\boldsymbol{y})\,dS_{\boldsymbol{y}}<Cr^{-1},\ r\to\infty$.

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[2] This normalization can be achieved only in the case $\boldsymbol{u}_0\neq 0$. If $\boldsymbol{u}_0=0$, the behavior of the fundamental solution tensor at infinity is quite different from that of the tensor we consider. Our proof, however, remains valid after modification of minor details, and provides a new demonstration, in some respects simpler than that of LERAY, that his solution $\boldsymbol{u}(\boldsymbol{x})$ tends to zero at infinity. The rapid decay of the components of $\mathbf{X}$ outside the limited "wake" region in the case $\boldsymbol{u}_0\neq 0$ is an essential feature of the present paper.

From (3) one concludes, by methods analogous to the usual methods of potential theory (*cf.* [2]), that for a suitable sequence of values $N \to \infty$, $\boldsymbol{u}_N(\boldsymbol{x})$, together with its derivatives of first order, converges uniformly in any fixed compact subregion of $\mathscr{E} \cup \Sigma$. As $N \to \infty$ along this sequence, the integral over Σ in (11) tends to a solution of (8) and by (12), this solution $\to 0$ as $r \to \infty$. It will be shown below that the integral over $\mathscr{E}_N$ in (11) is absolutely convergent, uniformly in N. For fixed $\boldsymbol{x}$, it tends to a limit as $N \to \infty$ along a suitable subsequence, but this limit may not be the integral of the corresponding expression involving the limit vector. However, the integral of the limit vector exists and tends to zero as $\boldsymbol{x} \to \infty$, and we shall be able to use this fact. To do so, it will be helpful to write (11) in a more tractable form. Observe that the fundamental solution tensor is not unique, and remains a fundamental solution tensor under the addition of an arbitrary solution tensor. The integral representation (11) thus remains valid if we replace the tensor $\mathbf{X}(\boldsymbol{x}, \boldsymbol{y})$ by the tensor

$$\bar{\mathbf{X}}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{X}(\boldsymbol{x}, \boldsymbol{y}) - \mathbf{X}(\mathbf{0}, \boldsymbol{y}) \tag{13}$$

provided the origin of coordinates is chosen interior to one of the finite regions bounded by Σ. The tensor $\bar{\mathbf{X}}$ has the advantage that certain types of singular behavior which may occur in integrals involving $\mathbf{X}(\boldsymbol{x}, \boldsymbol{y})$ at large distances from Σ are cancelled by the corresponding integrals involving $\mathbf{X}(\mathbf{0}, \boldsymbol{y})$.

A further difficulty appears in the study of the integral over Σ_N. Our procedure here will be to obtain a preliminary estimate using known information about the other terms in (11), and then to deform a modification of this integral into a suitable volume integral in such a way as to permit a satisfactory *a priori* estimate as $N \to \infty$.

3. Preliminary Lemmas

In order not to interrupt the continuity of the proof, we establish here some elementary preliminary results which we shall apply later.

Lemma 1. *Let* $\boldsymbol{u}(\boldsymbol{x})$ *be continuously differentiable in the closure of* $\mathscr{E}_N$, $\boldsymbol{u}(\boldsymbol{x}) = 0$ *on* Σ_N, *and suppose* $\int_{\mathscr{E}_N} |\nabla \boldsymbol{u}|^2 dV < K$. *Then there is a constant* K_1, *depending only on* K *(and not on* $\boldsymbol{x}$ *or on* N*), such that*

$$\int_{\mathscr{E}_N} \frac{\boldsymbol{u}^2}{(\boldsymbol{x} - \boldsymbol{y})^2} \, dV_{\boldsymbol{y}} < K_1 .$$

The inequality is essentially that of POINCARÉ [6]. The proof we give is due to LERAY [2]. Under the assumptions, we may extend the region of definition of the vectors $\boldsymbol{u}(\boldsymbol{x})$ to the entire interiors S_N of Σ_N, in such a way that the Dirichlet integral remains less than a fixed bound. From the identity

$$\int_{S_N} f \Delta g \, dV = -\int_{S_N} (\nabla f \cdot \nabla g) \, dV + \int_{\Sigma_N} f \frac{\partial g}{\partial n} \, dS$$

with $f = \boldsymbol{u}^2 = u_i^2$, $g = \frac{1}{2} \log (\boldsymbol{x} - \boldsymbol{y})^2 = \frac{1}{2} \log r^2$, we find

$$\int_{S_N} \frac{\boldsymbol{u}^2}{r^2} \, dV = -2 \int_{S_N} \frac{u_i \dfrac{\partial u_i}{\partial x_j} x_j}{r^2} \, dV \leqq 2K^{\frac{1}{2}} \left[\int_{S_N} \frac{\boldsymbol{u}^2}{r^2} \, dV \right]^{\frac{1}{2}} \tag{14}$$

from which the result follows directly.

Lemma 2. *Let* $\boldsymbol{v}(\boldsymbol{x})$ *denote a solution of the auxiliary system (equations of* Stokes)

$$\Delta \boldsymbol{v} - \nabla p = 0, \qquad \operatorname{div} \boldsymbol{v} = 0 \tag{15}$$

in the interior of a sphere Σ_N *of radius* N, *and suppose that on* Σ_N, $|\boldsymbol{v}| < KN^{-\alpha+\frac{1}{2}}$, $|\boldsymbol{v}_t| < KN^{-\alpha-\frac{1}{2}}$, *where* $\boldsymbol{v}_t$ *denotes an arbitrary derivative of* $\boldsymbol{v}$ *in a direction tangential to the surface* Σ_N. *Then* $|\boldsymbol{v}(\boldsymbol{x})| < KN^{-\alpha+\frac{1}{2}}$ *throughout the interior* S_N *of* Σ_N. *Interior to a concentric sphere of radius* $N/2$, $|\nabla \boldsymbol{v}| + |p| < KN^{-\alpha-\frac{1}{2}}$, *provided* p *is normalized to be (say) zero at the origin.*

Proof. The vector $\bar{\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{v}(N\boldsymbol{x})$ defines a solution of (15) in the unit sphere Σ_1. Consider the tensor and vector defined by

$$\mathcal{G}'_{ij} = \frac{1}{8\pi}\left[\frac{\delta_{ij}}{r} + \frac{(y_i - x_i)\,(y_j - x_j)}{r^3}\right],$$

$$\mathcal{P}'_j = \frac{1}{4\pi}\left[\frac{y_j - x_j}{r^3}\right], \qquad r = |\boldsymbol{x} - \boldsymbol{y}|.$$

Let $\mathcal{G}''_{ij}$ denote the tensor whose column vectors, as functions of $\boldsymbol{y}$, define solutions of (15) interior to Σ_1 and are equal to the column vectors of $\mathcal{G}'_{ij}$ on the boundary. The existence of $\mathcal{G}''_{ij}$ is assured by the work of Odqvist [*1*]. Let $\mathcal{P}''_j$ denote the corresponding pressures. The "Green's tensor" $\boldsymbol{\mathcal{G}}(\boldsymbol{x}, \boldsymbol{y})$ determined by

$$\mathcal{G}_{ij} = \mathcal{G}'_{ij} - \mathcal{G}''_{ij}, \qquad \mathcal{P}_j = \mathcal{P}'_j - \mathcal{P}''_j \tag{16}$$

then defines a fundamental solution tensor in Σ_1 such that each component $\mathcal{G}_{ij}$ vanishes on the boundary[3]. We have the estimate of Odqvist, $|T\boldsymbol{\mathcal{G}}| < Kr^{-2}$ uniformly in the closure of the sphere.

The representation (6b) gives

$$\bar{\boldsymbol{v}}(\boldsymbol{x}) = \int_{\Sigma_1} \boldsymbol{v}(N\boldsymbol{y})\; T\boldsymbol{\mathcal{G}}\, dS.$$

For a point $\boldsymbol{x} \neq 0$, we find

$$\bar{v}(\boldsymbol{x}) - \bar{v}\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) = \int_{\Sigma_1} \left[\boldsymbol{v}\cdot(N\boldsymbol{y}) - \boldsymbol{v}\left(N\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\right]\cdot T\boldsymbol{\mathcal{G}}\, dS$$

since every constant vector is again a solution of (15). We now divide Σ_1 into a hemisphere σ_1 which contains the point $\boldsymbol{x}/|\boldsymbol{x}|$ and is symmetric with respect to this point, and into the remaining hemisphere $\bar{\sigma}_1$. We find, in a suitable system of spherical coordinates,

$$\left|\bar{\boldsymbol{v}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\right| \leqq \frac{KN}{N^{\alpha+1/2}} \int_{\sigma_1} \varphi\, |T\boldsymbol{\mathcal{G}}| \sin\varphi\, d\varphi\, d\vartheta + \left[\frac{K}{N^{2\alpha-1}} \int_{\bar{\sigma}_1} dS \int_{\bar{\sigma}_1} |T\boldsymbol{\mathcal{G}}|^2\, dS\right]^{\frac{1}{2}},$$

where we have used the mean value theorem to estimate $\left[\boldsymbol{v}(N\boldsymbol{y}) - \boldsymbol{v}\left(N\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\right]$. But $|T\boldsymbol{\mathcal{G}}| < \frac{K}{|\boldsymbol{x}-\boldsymbol{y}|^2}$, and $|\sin\varphi| \leqq |\boldsymbol{x} - \boldsymbol{y}|$. Hence, the first term on the right is bounded by $KN^{1/2-\alpha}$. The second term is clearly bounded by $KN^{1/2-\alpha}$, and the result on $\boldsymbol{v}$ follows directly. The estimate on $|\nabla \boldsymbol{v}| + |p|$ is obtained immediately from the representation, since $T\mathcal{G}$ remains uniformly bounded when $\boldsymbol{x}$ lies interior to a sphere of radius $\frac{1}{2}$.

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[3] This tensor has been determined explicitly by Oseen ([*3*], p. 105). The estimate $|T\boldsymbol{\mathcal{G}}| < Kr^{-2}$ can be based alternatively on the formula of Oseen.

Lemma 3. *Let $\boldsymbol{F}(\boldsymbol{\xi})$ denote a vector-valued function which is defined for all points $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ which lie in an open connected set and have rational coordinates, and suppose that for some integer $n > 0$, all difference quotients $\Delta^{(n)} \boldsymbol{F}(\boldsymbol{\xi})$ of order n vanish identically for all differences taken with respect to such points. Then $\boldsymbol{F}(\boldsymbol{\xi})$ is a polynomial of degree at most $n-1$,*

$$\boldsymbol{F}(\boldsymbol{\xi}) = \sum_{i+j+k=n-1} \mathbf{c}_{ijk}\, \xi_1^i \xi_2^j \xi_3^k + \cdots, \tag{17}$$

where the $\mathbf{c}_{ijk}$ are vector-valued constants, and the terms not indicated explicitly constitute a polynomial of degree $n-2$ at most.

Proof. It clearly suffices to prove the result for a cube contained in the region of definition and with faces orthogonal to the coordinate axes. Consider in such a cube a line segment L: $a \leqq \xi_1 \leqq b$, $\xi_2, \xi_3 =$ rational const., and choose on this segment n equally spaced points $\boldsymbol{x}_0, \boldsymbol{x}_1 = \boldsymbol{x}_0 + h, \ldots, \boldsymbol{x}_{n-1} = \boldsymbol{x}_0 + nh$, having rational coordinates. Denote by $\boldsymbol{P}(\boldsymbol{t})$ the polynomial of degree $n-1$ such that $\boldsymbol{P}(\boldsymbol{t}) = \boldsymbol{F}(\boldsymbol{t})$ for $\boldsymbol{t} = \boldsymbol{x}_0, \ldots, \boldsymbol{x}_{n-1}$. On L set $\boldsymbol{G}(\boldsymbol{t}) = \boldsymbol{F} - \boldsymbol{P}$. Then $\boldsymbol{G}(\boldsymbol{t}) = 0$ for $\boldsymbol{t} = \boldsymbol{x}_0, \ldots, \boldsymbol{x}_{n-1}$, and $\Delta^n \boldsymbol{G}(\boldsymbol{t}) = 0$ for all difference quotients of order n taken with respect to rational values of $\boldsymbol{t}$ on L. If $\boldsymbol{G}(\boldsymbol{t}) \neq 0$, then there is a rational point $\boldsymbol{x}_n$ on L such that $\boldsymbol{G}(\boldsymbol{x}_n) \neq 0$. Choose $\boldsymbol{t} = \boldsymbol{x}_0$ and note that for this $\boldsymbol{t}$ successive difference quotients of order $\leqq n-1$ and increment h vanish, since $\boldsymbol{G}(\boldsymbol{t})$ then vanishes at all points of evaluation. Using the point $\boldsymbol{x}_n$ to form a final difference we find that for this choice, $\Delta^{(n)} G \neq 0$, a contradiction. Hence $\boldsymbol{G}(\boldsymbol{t}) = 0$, and we conclude that $\boldsymbol{F}(\boldsymbol{\xi})$ is a polynomial in ξ_1 of degree not more than $n-1$, the coefficients of which may be functions of ξ_2, ξ_3. Consider now the coefficient of ξ_1^{n-1}. If there were a difference quotient of this coefficient which did not vanish, it would again be possible to form a non-vanishing quotient of $\boldsymbol{F}(\boldsymbol{\xi})$ of order n. Hence this coefficient is constant. Proceeding in this way we obtain after a finite number of steps the required form for $\boldsymbol{F}(\boldsymbol{\xi})$, q.e.d.

4. Proof of Theorem 1

For the annular region bounded by Σ and by Σ_N, we have from (11) and (6) the integral representation

$$\begin{aligned} \boldsymbol{u}_N(\boldsymbol{x}) = &-\int_{\mathcal{E}_N} \overline{\mathbf{X}} \cdot (\boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N)\, dV + \\ &+ \int_{\Sigma} (\overline{\mathbf{X}} \cdot T\boldsymbol{u}_N - \boldsymbol{u}_N \cdot T\overline{\mathbf{X}} - \boldsymbol{u}_N \cdot \overline{\mathbf{X}} n_1)\, dS + \int_{\Sigma_N} \overline{\mathbf{X}} \cdot T\boldsymbol{u}_N\, dS, \end{aligned} \tag{18}$$

where $\mathbf{X}$ is defined by (13). We know from the results of LERAY that $\int_{\mathcal{E}_N} |\nabla \boldsymbol{u}_N|^2 dV < K$ independent of N, and that the sequence of integers N may be so chosen that $\boldsymbol{u}_N$ converges uniformly together with its derivatives of first order on Σ and in every compact subregion of $\mathcal{E}$ to a solution of (1′). We restrict attention to this sequence of integers and consider the limit function $\boldsymbol{u}(\boldsymbol{x})$. Clearly $\int_G |\nabla \boldsymbol{u}|^2 dV < K$ for any fixed compact subregion G, hence, $\int_{\mathcal{E}} |\nabla \boldsymbol{u}|^2 dV \leqq K$. Choose $\varepsilon > 0$. Choose

N_0 so large that $\int\limits_{\mathcal{E}-\mathcal{E}_{N_0}} |\nabla \boldsymbol{u}|^2 dV < \varepsilon^2$. Choose $R > N_0$ and so large that

(i) $\dfrac{1}{R^{\frac{1}{2}}} < \varepsilon$, and so that when $|\boldsymbol{x}| \geqq R$,

(ii) $\left| \int\limits_{\Sigma} (\boldsymbol{u} \cdot T\mathbf{X} - \mathbf{X} \cdot T\boldsymbol{u} - \mathbf{X} \cdot \boldsymbol{u})\, dS \right| < \varepsilon$ and

(iii) $\left| \int\limits_{\mathcal{E}_{N_0}} \mathbf{X} \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u})\, dV \right| < \varepsilon$.

These inequalities can be achieved because of the properties (12) of **X** and of **P**. Further, it is clear that for all N sufficiently large (i), (ii), and (iii) will be achieved also when $\boldsymbol{u}(\boldsymbol{x})$ is replaced by the approximating vectors $\boldsymbol{u}_N$.

We now consider a value of $\boldsymbol{x}$ such that $|\boldsymbol{x}| = R$, and write (18) in the form

$$\begin{aligned} \boldsymbol{u}_N(\boldsymbol{x}) = {} & \int\limits_{\mathcal{E}_{N_1}} \mathbf{X}(0, \boldsymbol{y}) \cdot (\boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N)\, dV_{\boldsymbol{y}} - \int\limits_{\mathcal{E}_{N_0}} \mathbf{X}(\boldsymbol{x}, \boldsymbol{y}) \cdot (\boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N)\, dV_{\boldsymbol{y}} - \\ & - \int\limits_{\mathcal{E}_{N_1}-\mathcal{E}_{N_0}} \mathbf{X}(\boldsymbol{x}, \boldsymbol{y}) \cdot (\boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N)\, dV_{\boldsymbol{y}} - \int\limits_{\mathcal{E}_N-\mathcal{E}_{N_1}} \overline{\mathbf{X}}(\boldsymbol{x}, \boldsymbol{y}) \cdot (\boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N)\, dV_{\boldsymbol{y}} + \\ & + \int\limits_{\Sigma} [\boldsymbol{u}_N \cdot T\mathbf{X}(0, \boldsymbol{y}) + \boldsymbol{u}_N \cdot \mathbf{X}(0, \boldsymbol{y})\, n_1 - \mathbf{X}(0, \boldsymbol{y}) \cdot T\boldsymbol{u}_N]\, dS_{\boldsymbol{y}} - \\ & - \int\limits_{\Sigma} [\boldsymbol{u}_N \cdot T\mathbf{X}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{u}_N \cdot \mathbf{X}(\boldsymbol{x}, \boldsymbol{y})\, n_1 - \mathbf{X}(\boldsymbol{x}, \boldsymbol{y}) \cdot T\boldsymbol{u}_N]\, dS_{\boldsymbol{y}} + \\ & + \int\limits_{\Sigma_N} \overline{\mathbf{X}} \cdot T\boldsymbol{u}_N\, dS_{\boldsymbol{y}} \\ = {} & \text{①} + \text{②} + \text{③} + \text{④} + \text{⑤} + \text{⑥} + \text{⑦}, \end{aligned} \tag{19}$$

where we have chosen $N_1 = R^3$. With the above choice of N_0, R, N_1 we have $|\text{①} - C_1| < K\varepsilon$, $C_1 = \int\limits_{\mathcal{E}} \mathbf{X}(0, \boldsymbol{y}) \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u})\, dV_{\boldsymbol{y}}$. This integral is absolutely convergent, for it is dominated by $\left[\int\limits_{\mathcal{E}} \mathbf{X}^2(0, \boldsymbol{y})\, \boldsymbol{u}^2(\boldsymbol{y})\, dV_{\boldsymbol{y}} \int\limits_{\mathcal{E}} |\nabla \boldsymbol{u}|^2 dV_{\boldsymbol{y}} \right]^{\frac{1}{2}}$ and the estimate follows directly from Lemma 1, from (12), and from the choice of N_0 and $N_1 > N_0$. We have also $|\text{②}| < \varepsilon$ by (iii) above. Further,

$$|\text{③}|^2 \leqq \int\limits_{\mathcal{E}_N} \mathbf{X}^2 \boldsymbol{u}_N^2\, dV_{\boldsymbol{y}} \int\limits_{\mathcal{E}_{N_1}-\mathcal{E}_{N_0}} |\nabla \boldsymbol{u}_N|^2 dV_{\boldsymbol{y}} \leqq K\varepsilon^2$$

for all sufficiently large N, by Lemma 1, (12), and the choice of N_0. To study ④, observe that for any component of $\overline{\mathbf{X}}$, $|\overline{\mathrm{X}}| = |\mathrm{X}(\boldsymbol{x}, \boldsymbol{y}) - \mathrm{X}(0, \boldsymbol{y})| = |\mathrm{X}'(\boldsymbol{\xi}, \boldsymbol{y})|\, R$, where $\boldsymbol{\xi}$ denotes a point on the segment joining the origin to the point $\boldsymbol{x}$, and the derivative $\mathbf{X}'$ is taken in the direction of this segment. Thus, for $\boldsymbol{y} \in \mathcal{E} - \mathcal{E}_{N_1}$, $N_1 = R^3$, we have, using (12),

$$|\overline{\mathbf{X}}| \leqq \frac{KR}{r^{\frac{3}{2}}} \leqq \frac{K}{R^{\frac{1}{2}} r} \leqq \frac{K\varepsilon}{r}$$

by the property (i) of R above, where $r = |\boldsymbol{x} - \boldsymbol{y}|$. Hence,

$$|\text{④}|^2 \leqq K^2 \varepsilon^2 \int\limits_{\mathcal{E}_N} \frac{\boldsymbol{u}_N^2}{r^2}\, dV_{\boldsymbol{y}} \int\limits_{\mathcal{E}_N} |\nabla \boldsymbol{u}_N|^2 dV_{\boldsymbol{y}} \leqq K^2 \varepsilon^2$$

for all $N > N_1$.

Clearly ⑤ tends to a constant, ⑤ $\to C_2$, as $N\to\infty$. By the choice of R, $|\text{⑥}|<\varepsilon$. Therefore,

$$\boldsymbol{u}_N(\boldsymbol{x}) = \boldsymbol{C} + \boldsymbol{\epsilon}_N(\boldsymbol{x}) + \text{⑦} \tag{20}$$

where $\boldsymbol{\epsilon}_N(\boldsymbol{x})$ is such that $\overline{\lim_{N\to\infty}}\, |\boldsymbol{\epsilon}_N(\boldsymbol{x})|$ tends to zero as $R\to\infty$.

Now let $N\to\infty$ for fixed $\boldsymbol{x}$. The left side of (20) tends to a limit, and all quantities on the right except ⑦ have been shown to remain bounded. Hence, also ⑦ is bounded. We may clearly choose a subsequence of values $N\to\infty$ such that for any point $\boldsymbol{x}$ with rational coordinates, all quantities in (20) tend to finite limits as $N\to\infty$. For this sequence $\{N\}$ we obtain a function

$$\boldsymbol{F}(\boldsymbol{x}) = \lim_{N\to\infty} \int_{\Sigma_N} \overline{\mathbf{X}}(\boldsymbol{x},\boldsymbol{y})\cdot T\boldsymbol{u}_N(\boldsymbol{y})\, dS_{\boldsymbol{y}}$$

defined and finite for all rational $\boldsymbol{x}$ of $\mathcal{E}$. Corresponding to arbitrary rational differences, we may form the difference quotients

$$\Delta^{(n)}\boldsymbol{F}(\boldsymbol{x}) = \lim_{N\to\infty} \int_{\Sigma_N} \Delta^{(n)}\overline{\mathbf{X}}(\boldsymbol{x},\boldsymbol{y})\cdot T\boldsymbol{u}_N(\boldsymbol{y})\, dS_y .$$

Since the row and column vectors of $\overline{\mathbf{X}}$ are divergence free, and since the elements of $\overline{\mathbf{X}}$ are majorized by C/r in the neighborhood of the singular point, we have

$$\int_{\Sigma_N} \Delta^{(n)}\overline{\mathbf{X}}\cdot d\boldsymbol{S}_{\boldsymbol{y}} = 0 .$$

Hence, by the work of ODQVIST [*1*] we are assured of the existence interior to Σ_N of a solution $\boldsymbol{v}_N^{(n)}$ of the Stokes equations (15) such that $\boldsymbol{v}_N^{(n)} = \Delta^{(n)}\overline{\mathbf{X}}$ on Σ_N. Since $\boldsymbol{u}_N(\boldsymbol{x}) = 0$ on Σ_N, the identity (6b) gives

$$\int_{\Sigma_N} \Delta^{(n)}\overline{\mathbf{X}}\cdot T\boldsymbol{u}_N\, dS = \int_{\mathcal{E}_N} \boldsymbol{v}_N^{(n)}\cdot(\boldsymbol{u}_N\cdot\nabla\boldsymbol{u}_N)\, dV + \int_{\mathcal{E}_N} \boldsymbol{v}_N^{(n)}\cdot\frac{\partial\boldsymbol{u}_N}{\partial x_1}\, dV + \\ + \int_{\Sigma} [\boldsymbol{u}_N\cdot T\boldsymbol{v}_N^{(n)} - \boldsymbol{v}_N^{(n)}\cdot T\boldsymbol{u}_N]\, dS .$$

From Lemma 2 follows directly that the integral over Σ tends to zero as $N\to\infty$ for any $n\geqq 0$. But

$$\left[\int_{\mathcal{E}_N} \boldsymbol{v}_N^{(n)}\cdot(\boldsymbol{u}_N\cdot\nabla\boldsymbol{u}_N)\, dV\right]^2 \leqq K\int_{\mathcal{E}_N} (\boldsymbol{v}_N^{(n)})^2\, \boldsymbol{u}_N^2\, dV,$$

$$\left[\int_{\mathcal{E}_N} \boldsymbol{v}_N^{(n)}\cdot\frac{\partial\boldsymbol{u}}{\partial x_1}\, dV\right]^2 \leqq K\int_{\mathcal{E}_N} (\boldsymbol{v}_N^{(n)})^2\, dV,$$

by the Schwarz inequality. By Lemma 2 and the property (12 iii) of $\mathbf{X}$, if n is sufficiently large, the right hand integrals tend to zero as $N\to\infty$. (For our case it is sufficient to choose $n\geqq 2$.) Hence, for all n larger than this fixed integer, we have $\Delta^{(n)}\boldsymbol{F}(\boldsymbol{x}) = 0$ at all rational points x of $\mathcal{E}$. Applying Lemma 3, we see that $\boldsymbol{F}(\boldsymbol{x})$ is at the rational points a polynomial $\boldsymbol{P}(\boldsymbol{x})$ in x_1, x_2, x_3. But $\boldsymbol{u}_N(\boldsymbol{x})\to\boldsymbol{u}(\boldsymbol{x})$ at *all* points exterior to Σ. Thus, from (20) and from the continuity of $\boldsymbol{u}(\boldsymbol{x})$ we obtain that for *any* $\boldsymbol{x}$, the limit function $\boldsymbol{u}(\boldsymbol{x})$ differs from $\boldsymbol{P}(\boldsymbol{x})+\boldsymbol{C}$ by at most a quantity which becomes arbitrarily small as $\boldsymbol{x}\to\infty$. On the other hand, we know that

$$\int\limits_{\mathscr{E}} \frac{\boldsymbol{u}^2}{r^2}\, dV < \infty$$

(Lemma 1), and this can only be if $\boldsymbol{P}(\boldsymbol{x}) + \boldsymbol{C} \equiv 0$. Thus, $\lim\limits_{\boldsymbol{x}\to\infty} \boldsymbol{u}(\boldsymbol{x}) = 0$, *i.e.*, the velocity vector tends to its prescribed limit at infinity.

To prove (4), choose

$$R_1 \geqq R^3 \qquad (\text{for } |\boldsymbol{x}| = R),$$

and observe that we can now write (18) in the form

$$\left| \boldsymbol{u}_N(\boldsymbol{x}) + \int\limits_{\mathscr{E}_{R_1}} \mathbf{X} \cdot (\boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N)\, dV - \int\limits_{\Sigma} (\mathbf{X} \cdot T\boldsymbol{u}_N - \boldsymbol{u}_N \cdot T\mathbf{X} - \boldsymbol{u}_N \cdot \mathbf{X}\, n_1)\, dS \right| \leqq \varepsilon_N(R_1),$$

where $\overline{\lim\limits_{N\to\infty}}\, \varepsilon_N(R_1) \to 0$ as $R_1 \to \infty$. The desired representation follows by letting $N \to \infty$, then $R_1 \to \infty$, and reverting to original notation. To obtain (4′) we begin with the representation (*cf.* [3])

$$p_N(\boldsymbol{x}) = -\int\limits_{\mathscr{E}_N} \boldsymbol{\psi} \cdot (\boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N)\, dV + \int\limits_{\Sigma+\Sigma_N} \{\boldsymbol{\psi} \cdot T\boldsymbol{u}_N - \boldsymbol{u}_N \cdot T\boldsymbol{\psi} - \boldsymbol{u}_N \cdot \boldsymbol{\psi}\, n_1\}\, dS,$$

where we have set $\boldsymbol{\psi} = \nabla\left(\frac{1}{r}\right)$, $r = |\boldsymbol{x} - \boldsymbol{y}|$ and introduced a corresponding pressure $q = -\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)$. To estimate the volume integral, we again choose N_0 so that $\int\limits_{\mathscr{E}-\mathscr{E}_{N_0}} |\nabla \boldsymbol{u}|^2\, dV < \varepsilon$, and then choose R so large that for $|\boldsymbol{x}| \geqq R$, $\left| \int\limits_{\mathscr{E}_{N_0}} \boldsymbol{\psi}\, (\boldsymbol{u} \cdot \nabla \boldsymbol{u})\, dV \right| < \varepsilon$. In the region $\mathscr{E}_N - \mathscr{E}_{N_1}$, $N_1 = 2R$, we have

$$\left| \int \boldsymbol{\psi} \cdot (\boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N)\, dV \right|^2 \leqq \frac{K}{R^2} \int\limits_{\mathscr{E}} \frac{\boldsymbol{u}_N^2}{r^2}\, dV \int\limits_{\mathscr{E}} |\nabla \boldsymbol{u}_N)^2\, dV$$

which $\to 0$ as $R \to \infty$ uniformly in N (N sufficiently large). We cannot estimate the remaining integral over $\mathscr{E}_{N_1} - \mathscr{E}_{N_0}$ in this way because of the large singularity of $\boldsymbol{\psi}$ at $r = 0$. But we have already proved that $\boldsymbol{u}(\boldsymbol{x}) \to 0$ as $\boldsymbol{x} \to \infty$, and from this follows easily that $\nabla \boldsymbol{u}(\boldsymbol{x}) \to 0$ as $\boldsymbol{x} \to \infty$ (*cf.* [2]). Thus, for sufficiently large $|\boldsymbol{x}|$, we shall have $|\boldsymbol{u}_N(\boldsymbol{x})|$ and $|\nabla \boldsymbol{u}_N(\boldsymbol{x})|$ as small as we wish provided only that N is large enough (depending on $|\boldsymbol{x}|$). Hence, by choosing R suitably large and considering a unit sphere S_1 centered at $\boldsymbol{x}$, $|\boldsymbol{x}| = R$, we can have

$$\left| \int\limits_{S_1} \boldsymbol{\psi}\, (\boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N)\, dV \right| < \varepsilon$$

for all sufficiently large N. Finally,

$$\begin{aligned} \left| \int\limits_{\mathscr{E}_{N_1}-\mathscr{E}_{N_0}-S_1} \boldsymbol{\psi}\, (\boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N)\, dV \right|^2 &\leqq K \int\limits_{\mathscr{E}} \frac{\boldsymbol{u}_N}{r^2}\, dV \int\limits_{\mathscr{E}-\mathscr{E}_{N_0}} |\nabla \boldsymbol{u}|^2\, dV \\ &\leqq K\varepsilon \end{aligned}$$

by the choice of N_0, for all large enough N. It follows that the volume integral becomes arbitrarily small for large $|\boldsymbol{x}|$, uniformly in N as $N \to \infty$.

The corresponding estimate for the integrals over Σ is obvious.

All integrals over Σ_N tend to zero as $N\to\infty$, $\boldsymbol{x}$ fixed, except possibly $\int\limits_{\Sigma_N}\boldsymbol{\psi}\cdot T\boldsymbol{u}_N\,dS$. Hence, this integral remains bounded as $N\to\infty$, and we can choose a subsequence of the $\{N\}$ so that

$$F(\boldsymbol{x})=\lim_{N\to\infty}\int\limits_{\Sigma_N}\boldsymbol{\psi}\cdot T\boldsymbol{u}_N\,dS$$

is defined and finite at all rational points. Taking a *single* difference quotient $\Delta^{(1)}F(\boldsymbol{x})$ and repeating the previous procedure of replacing the difference quotient $\Delta^{(1)}\boldsymbol{\psi}$ by the solution $\boldsymbol{v}_N^{(1)}$ of (15) equal to $\Delta^{(1)}\boldsymbol{\psi}$ on Σ_N, we obtain an estimate[4] on $\boldsymbol{v}_N^{(1)}$ sufficiently good to ensure that $\Delta^{(1)}F(\boldsymbol{x})=0$, *i.e.*, $F(\boldsymbol{x})=\text{const.}=p_0$. Inserting this constant in the above representation for $p(\boldsymbol{x})$ gives (4′), and completes the proof of Theorem 1.

5. Proof of Theorem 2. Alternative Proof of Theorem 1

We base the proof of Theorem 2 on a lemma of PAYNE & WEINBERGER [*4*], concerning integral means of functions with finite Dirichlet integrals in a neighborhood of infinity. We shall need, actually, a somewhat more general result, for which we provide a short proof.

Lemma 1′. *Let $\boldsymbol{u}(\boldsymbol{x})$ denote a vector-valued function with continuous first derivatives, defined in the exterior E_{R_0} of a sphere Σ_{R_0} of radius R_0, and suppose that the Dirichlet integral $\Delta_{R_0}[\boldsymbol{u}]$ extended over E_{R_0} is finite. Then*

i) *The integral means $\bar{\boldsymbol{u}}$ of $\boldsymbol{u}(\boldsymbol{x})$ over concentric spheres of radius R tend to a limit $\boldsymbol{u}_0$ as $R\to\infty$, i.e., if*

$$\bar{\boldsymbol{u}}(R)=\frac{1}{4\pi}\int\limits_{\Sigma_1}\boldsymbol{u}(R\boldsymbol{\omega})\,d\Omega,$$

where $\boldsymbol{\omega}$, $d\Omega$ denote a point and surface element on the unit sphere, respectively, then there exists a $\boldsymbol{u}_0$ such that $\lim\limits_{r\to\infty}\bar{\boldsymbol{u}}(R)=\boldsymbol{u}_0$.

ii) *For any $R\geqq R_0$,*

$$\frac{1}{R}\int\limits_{\Sigma_R}(\boldsymbol{u}-\boldsymbol{u}_0)^2\,dS\leqq\Delta_R[\boldsymbol{u}]=\int\limits_{E_R}|\nabla\boldsymbol{u}|^2\,dV.$$

iii) *If $r_{\boldsymbol{x}\boldsymbol{y}}$ denotes the distance between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ of E_{R_0}, then*

$$\int\limits_{E_{R_0}}\frac{[\boldsymbol{u}(\boldsymbol{y})-\boldsymbol{u}_0]^2}{r_{\boldsymbol{x}\boldsymbol{y}}^2}\,dV_{\boldsymbol{y}}<K<\infty$$

independent of $\boldsymbol{x}$, where K depends only on $\Delta_{R_0}[\boldsymbol{u}]$.

The proofs of i) and ii) are given in [*4*] and will not be repeated here. To prove iii) we note that $\boldsymbol{u}(\boldsymbol{x})$ can be extended to the entire space as a continuous function with bounded Dirichlet integral, depending only on $\Delta_{R_0}[\boldsymbol{u}]$. Consider now a sphere S_R of radius R and center at $\boldsymbol{x}$. In this sphere the relation (14) becomes, noting that $\boldsymbol{u}(\boldsymbol{x})$ no longer need vanish on the surface Σ_R,

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[4] The magnitude of the vector field $\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y})$ decreases sufficiently rapidly at infinity that a formal manipulation would imply $F(\boldsymbol{x})=0$. However, this vector field has the character of a potential flow due to a source at the point $\boldsymbol{y}=\boldsymbol{x}$, and creates a net flux across the boundary of any region enclosing the source. Therefore, there cannot exist a non-singular, divergence-free vector field $\boldsymbol{v}_N(\boldsymbol{x})$ equal to $\boldsymbol{\psi}(\boldsymbol{x})$ on the boundary. This behavior must be expected, for the pressure is defined by a solution of (1) only up to an additive constant.

$$\int\limits_{S_R} \frac{(\boldsymbol{u}-\boldsymbol{u}_0)^2}{r^2}\, dV = -2 \int\limits_{S_R} (u_i - u_{0i}) \frac{\partial u_i}{\partial x_j} \frac{x_j}{r^2}\, dV + \frac{1}{R} \int\limits_{\Sigma_R} (\boldsymbol{u}-\boldsymbol{u}_0)^2\, dS,$$

and the result follows directly from Schwarz' inequality and Lemma 1′, ii).

To prove Theorem 2, we consider, as before, the annular region $\mathscr{E}_N$ bounded by a sphere S_N and Σ. We assume again[5] that $\boldsymbol{u}_0 = (1, 0, 0)$, and we write the equations in terms of the variable $\boldsymbol{u}-\boldsymbol{u}_0$, which, changing notation as before, we write simply as $\boldsymbol{u}(\boldsymbol{x})$. We note that since we are dealing with a single vector field rather than a sequence, we shall not be bothered by questions of uniform convergence, and it is therefore no longer necessary to introduce the tensor $\mathbf{X}(\mathbf{0}, \boldsymbol{y})$. We obtain the integral representation

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{x}) = \int_{\mathscr{E}_N} \mathbf{X} \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u})\, dV + \int_{\Sigma} (\mathbf{X} \cdot T\boldsymbol{u} - \boldsymbol{u} \cdot T\mathbf{X} - \boldsymbol{u} \cdot \mathbf{X} n_1)\, dS +\\ + \int_{\Sigma_N} (\mathbf{X} \cdot T\boldsymbol{u} - \boldsymbol{u} \cdot T\mathbf{X} - \boldsymbol{u} \cdot \mathbf{X} n_1)\, dS.\end{aligned}$$

Certain new integrals over Σ_N appear here, which did not occur in (18), since $\boldsymbol{u}(\boldsymbol{x})$ need no longer vanish on this surface.

The integral over Σ obviously tends to zero as $\boldsymbol{x} \to \infty$. To study the integral over $\mathscr{E}_N$, we observe that for any $\varepsilon > 0$, we may choose N_0 so large that $\int_{\mathscr{E}-\mathscr{E}_{N_0}} |\nabla \boldsymbol{u}|^2\, dV < \varepsilon^2$, and then $|\boldsymbol{x}|$ so large that $\left| \int_{\mathscr{E}_{N_0}} \mathbf{X} \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u})\, dV \right| < \varepsilon$. For this $\boldsymbol{x}$,

$$\left| \int_{\mathscr{E}_N - \mathscr{E}_{N_0}} \mathbf{X} \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{v})\, dV \right|^2 \leqq \int_{\mathscr{E}} \mathbf{X}^2 \boldsymbol{u}^2\, dV \int_{\mathscr{E}-\mathscr{E}_{N_0}} |\nabla \boldsymbol{u}|^2\, dV$$

and hence, by Lemma 1′ and the properties (12) of $\mathbf{X}$, this integral tends to zero as $\boldsymbol{x} \to \infty$.

We are left with three integrals over Σ_N. Two of them can be estimated directly, for

$$\left| \int\limits_{\Sigma_N} \boldsymbol{u} \cdot T\mathbf{X}\, dS \right|^2 \leqq K \int\limits_{\Sigma_N} \frac{\boldsymbol{u}^2}{N}\, dS \int\limits_{\Sigma_N} N^{-2}\, dS,$$

$$\left| \int\limits_{\Sigma_N} \boldsymbol{u} \cdot \mathbf{X} n_1\, dS \right| \leqq N \int\limits_{\Sigma_N} \frac{\boldsymbol{u}^2}{N}\, dS \int\limits_{\Sigma_N} \mathbf{X}^2\, dS.$$

Each of these terms tends to zero as $N \to \infty$, because of (12), Lemma 1′, and the fact that the Dirichlet integral of $\boldsymbol{u}(\boldsymbol{x})$ exterior to Σ_N becomes arbitrarily small if N is large enough.

We estimate the final term exactly as we have estimated the corresponding term in the proof of Theorem 1, a simplification appearing in the fact that the limit of this integral is now seen to yield a function

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[5] The exceptional case $\boldsymbol{u}_0 = 0$ requires only minor technical changes in the proof; *cf.* footnote [2].

$$\boldsymbol{F}(\boldsymbol{x}) = \lim_{N\to\infty} \int_{\Sigma_N} \mathbf{X} \cdot T\boldsymbol{u}\, dS$$

which is defined and continuous at all points, not just the rational points. Again forming difference quotients and replacing $\Delta^{(n)}\mathbf{X}$ interior to Σ_N by the solution $\boldsymbol{v}_N^{(n)}$ of (15) which equals $\Delta^{(n)}\mathbf{X}$ on Σ_N, we find

$$\int_{\Sigma_N} \Delta^{(n)}\mathbf{X} \cdot T\boldsymbol{u}\, dS = \int_{\mathscr{E}_N} \boldsymbol{v}_N^{(n)} \cdot \left(\frac{\partial \boldsymbol{u}}{\partial x_1} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) dV + \\ + \int_{\Sigma} [\boldsymbol{u} \cdot T\boldsymbol{v}_N^{(n)} - \boldsymbol{v}_N^{(n)} \cdot T\boldsymbol{u}]\, dS + \int_{\Sigma_N} \boldsymbol{u} \cdot T\boldsymbol{v}_N^{(n)}\, dS.$$

The estimates proceed as in the study of (19), ⑦, except for the appearance of a new boundary integral $\int_{\Sigma_N} \boldsymbol{u} \cdot T\boldsymbol{v}_N^{(n)}\, dS$. It follows from Lemma 1′ that this integral tends to zero as $N\to\infty$, provided we know that for n sufficiently large $|T\boldsymbol{v}_N^{(n)}| < KN^{-\frac{3}{2}}$ on Σ_N. The proof of this fact can be obtained in a manner entirely analogous to the proof of the corresponding estimate of potential theory, using the Green's tensor (16) in place of the Green's function of potential theory, and we omit the details.

We now apply Theorem 2 to obtain a second proof of Theorem 1. We observe, using the original notation, that the approximating solutions $\boldsymbol{u}_N(\boldsymbol{x})$ of LERAY tend to a solution $\boldsymbol{u}(\boldsymbol{x})$ in $\mathscr{E}$ for which $\int_{\mathscr{E}} \frac{[\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_0]^2}{r^2}\, dV < \infty$, and for which $\int_{\mathscr{E}} |\nabla \boldsymbol{u}|^2\, dV < \infty$. Hence, by Theorem 2, there exists a vector $\boldsymbol{u}_1$ such that $\boldsymbol{u}(\boldsymbol{x}) \to \boldsymbol{u}_1$ as $\boldsymbol{x}\to\infty$. It follows immediately that $\boldsymbol{u}_1 = \boldsymbol{u}_0$, *q.e.d.*

6. Proof of Theorem 3

Once more, we replace the symbol $\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_0$ by the symbol $\boldsymbol{u}(\boldsymbol{x})$. We are given $|\boldsymbol{u}(\boldsymbol{x})| < Cr^{-\alpha}$, $\alpha > \frac{1}{2}$. We have, by Theorem 2, the representation

$$\boldsymbol{u}(\boldsymbol{x}) = \int_{\Sigma} (\mathbf{X} \cdot T\boldsymbol{u} - \boldsymbol{u} \cdot T\mathbf{X} - \boldsymbol{u} \cdot \mathbf{X} n_1)\, dS - \int_{\mathscr{E}} \mathbf{X} \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u})\, dV.$$

Since $\operatorname{div} \boldsymbol{u} = 0$, $\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \operatorname{div} \boldsymbol{u}\boldsymbol{u}$, and we may integrate the volume integral by parts to obtain

$$\boldsymbol{u}(\boldsymbol{x}) = \int_{\Sigma} (\mathbf{X} \cdot T\boldsymbol{u} - \boldsymbol{u} \cdot T\mathbf{X} - \boldsymbol{u} \cdot \mathbf{X} n_1 - \mathbf{X} \cdot \boldsymbol{u}\boldsymbol{u})\, dS + \int_{\mathscr{E}} (\boldsymbol{u}\boldsymbol{u}) \cdot \nabla \mathbf{X}\, dV.$$

We have here estimated a boundary integral as in the proof of Theorem 2, and passed to a limit.

Using (12), we see that the integral over Σ has the order r^{-1} at infinity. To estimate the volume integral, we choose a point $\boldsymbol{x}$ with distance $2R$ from the origin (R sufficiently large), and divide the region of integration into three parts, as follows:

I: the annular region $\mathscr{E}_R$ bounded by Σ and by a sphere of radius R about the origin, II: the region G_R consisting of those points interior to a sphere of radius $3R$ about $\boldsymbol{x}$ as center, which do not lie in $\mathscr{E}_R$,

III: the region E_R of points whose distances from $\boldsymbol{x}$ exceed $3R$.

We have

$$\Big|\int\limits_{\mathscr{E}_R}(\boldsymbol{u}\boldsymbol{u})\cdot\nabla\mathbf{X}\,dV\Big|^2 \le \int\limits_{\mathscr{E}_R}\boldsymbol{u}^4\,dV\int\limits_{\mathscr{E}_R}|\nabla\mathbf{X}|^2\,dV$$

$$\le C(1+R)^{-4\alpha+3}\int\limits_{A_R}|\nabla\mathbf{X}|^2\,dV$$

by assumption, where A_R denotes the annular region bounded by spheres of radii R and $3R$ about $\boldsymbol{x}$. The integral over A_R can be estimated by a tedious but formal computation, using the definition (9) of $\mathbf{X}$ (it is necessary here to use a more careful estimate of $\mathbf{X}$ than is indicated in (12)). We obtain

$$\int\limits_{A_R}|\nabla\mathbf{X}|^2\,dV \le CR^{-2},$$

so that

$$\Big|\int\limits_{\mathscr{E}_R}(\boldsymbol{u}\boldsymbol{u})\cdot\nabla\mathbf{X}\,dV\Big| \le C(R^{-1}+R^{-2\alpha+\frac{1}{2}})$$

for a suitable constant C.

In G_R, we have everywhere $|\boldsymbol{u}|<CR^{-\alpha}$, so that

$$\Big|\int\limits_{G_R}(\boldsymbol{u}\boldsymbol{u})\cdot\nabla\mathbf{X}\,dV\Big| \le CR^{-2\alpha}\int\limits_{G_R}|\nabla\mathbf{X}|\,dV.$$

The integral on the right can again be estimated by formal computation, which yields

$$\Big|\int\limits_{G_R}(\boldsymbol{u}\boldsymbol{u})\cdot\nabla\mathbf{X}\,dV\Big| \le CR^{-2\alpha+\frac{1}{2}}.$$

Finally,

$$\Big|\int\limits_{E_R}(\boldsymbol{u}\boldsymbol{u})\cdot\nabla\mathbf{X}\,dV\Big| \le C\int\limits_{3R}^{\infty}r^{-2\alpha}\,dr\int\limits_{\Sigma_r}|\nabla\mathbf{X}|\,dS$$

$$\le CR^{-2\alpha+\frac{1}{2}}$$

by essentially the same calculation. Collecting these estimates, we see that $|\boldsymbol{u}(\boldsymbol{x})| \le C(R^{-2\alpha+\frac{1}{2}}+R^{-1})$ at infinity whenever $\alpha>\frac{1}{2}$. Set $\alpha=\frac{1}{2}+\varepsilon$, $\varepsilon>0$. We have proved that the inequality $|\boldsymbol{u}(\boldsymbol{x})| \le CR^{-\frac{1}{2}-\varepsilon}$ implies always the inequality $|\boldsymbol{u}(\boldsymbol{x})| \le C(R^{-\frac{1}{2}-\varepsilon}+R^{-1})$. A finite number of iterations then yields the desired estimate $|\boldsymbol{u}(\boldsymbol{x})| \le CR^{-1}$, *q.e.d.*

Remarks. The estimate of Theorem 3 is exactly the order of magnitude exhibited by the solutions of the linearized equations (8) in the "wake" region $r-x_1 \le C$. For paths tending to infinity outside this region a better estimate can be obtained, for contributions of the order r^{-1} appear only from the integral over the region I and the integral over Σ, and these contributions can be shown to be of smaller order when the point $\boldsymbol{x}$ lies outside the wake. The nature of the representation suggests also the possibility of examining the qualitative behavior of a one parameter family of solutions of (1) depending (say) on the viscosity, as the viscosity tends to a zero or infinite limit. The author hopes to return to these matters at a later time.

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A Study of the Interior Ballistic Equations

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Summary

A system of non-linear differential equations, typical of the interior ballistics equations for orthodox guns, is studied. Characteristics of the mathematical pressure curves are obtained for the entire spectrums of the ballistic parameter, burning exponent, and geometric parameter (characteristic of the form function). The mathematical solutions are found to distinguish sharply between regressive and progressive burning geometries, and a constant pressure solution is obtained (a piezometric efficiency of one) for what is defined as the ideally progressive geometry. The approximate validity of isothermal solutions, established by CLEMMOW for a linear burning law, is extended to the general case.

1. Introduction

As a technology, the interior ballistics of guns is in an advanced state. Indeed, a recent paper [*1*] suggests that the main problems have been almost completely solved. Still, throughout the many contributions to theory, little attention is given to mathematical aspects of the problems, and, as a result, the full impact of the equations is not always felt. Even in the most orthodox cases the distinction between practical ballistics and research ballistics is obscured by a common empirical flavor, and useful aspects of the mathematics appear to be lost in a zeal for correlation with experimental data. Thus, despite a long and continuous history, many fundamental questions concerning the gun equations have remained unanswered, and so perhaps a few purely mathematical considerations are appropriate notwithstanding the rather advanced state of the technology.

In the present paper we consider the interior ballistics problem as characterized by a system of non-linear differential equations. These equations are not unlike those appearing in numerous other investigations (*cf.* [*1*], [*2*], [*3*], [*4*], [*5*]). Here, however, we are concerned with the implications of the equations themselves, so that questions concerning the degree to which these equations or their solutions represent the physical situation are not considered. On the other hand, much of the mathematical study is motivated by practical aspects of the ballistics problem, and some of the results may prove useful either in design problems or as aids to an understanding of the fundamental ballistic processes. A list of nomenclature is provided as a ready link to practical applications of the results. By-and-large the results are new in the sense that they are here derived for the first time from the equations, although some may well reflect established technology. Several results, peculiar to propellants with progressive burning characteristics, are new in any sense and should prove both of practical and of theoretical interest.

Specifically the equations considered are introduced in § 2 and their solution interpreted geometrically in § 3. Characteristics of the pressure curves are derived in §§ 4 and 5, and parameter effects are studied §§ 6, 7, and 8. In § 9 we discuss the ideally progressive case which admits a constant pressure solution, and in § 10 we consider the significance of isothermal solutions.

2. The fundamental equations

The interior ballistics problem to be considered is in reference to the following equations*:

Momentum Equation

$$M\frac{d^2x}{dt^2} = AP. \tag{1}$$

Burning Law

$$\frac{df}{dt} = -\frac{B}{W}P^{\alpha}. \tag{2}$$

Equation of State

$$NF\frac{T}{T_0} = PAx. \tag{3}$$

Energy Equation

$$\frac{NF}{\gamma-1}\left(1-\frac{T}{T_0}\right) = \frac{1}{2}M\left(\frac{dx}{dt}\right)^2. \tag{4}$$

Form Function

$$N = C(1-f)(1+\vartheta f). \tag{5}$$

The principal nomenclature used is as follows:

A = Gun bore area
B = Propellant burning constant
C = Propellant charge
F = Propellant force
f = Fraction of propellant web remaining
M = Mass of shot
N = Fraction of charge burned
P = Instantaneous chamber pressure
P_a = Average pressure, defined by equation (24)
P_m = Mathematical maximum of P
$P_c = \frac{CF}{Ax_0}$ = Closed chamber pressure

$$\left.\begin{array}{l} P_1 = \left(\dfrac{B^2MFC}{W^2A^2}\right)^{\frac{1}{2-2\alpha}} \\ P_2 = \left(\dfrac{B^2MFC}{W^2A^2}\right)^{\frac{1}{2}} \\ P_3 = \left(\dfrac{B^2MF^2C^2}{W^2A^3x_0}\right)^{\frac{1}{3-2\alpha}} \end{array}\right\} \text{Ballistic parameters}$$

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* See [5, Chapter V], for example.

$r = \frac{1}{3-2\alpha}$

S = Surface, bounding trajectories, defined by equation (19)

T = Instantaneous gas temperature

T_0 = Adiabatic combustion temperature

t = Time

v = Instantaneous velocity (speed) of shot

W = Propellant web size

x = Shot travel

x_0 = Initial position of shot, defined as the initial chamber volume divided by the bore area A

α = Burning exponent, $0 \leqq \alpha \leqq 1$

$\beta = \frac{1}{2-\alpha}$

γ = Ratio of specific heats of propellant gases

ε_b = Instantaneous ballistic efficiency, defined by equation (75)

$\varepsilon_p = \frac{P_a}{P_m}$ = Instantaneous piezometric efficiency

ϑ = Propellant geometric parameter, $-1 \leqq \vartheta \leqq +1$, $-1 \leqq \vartheta < 0$ corresponds to progressive burning while $0 < \vartheta \leqq +1$ corresponds to regressive burning

$\mu = (1-f)^{2-\alpha}$

μ_m = Value of μ at maximum pressure

$\nu = \sqrt{\frac{M}{CF}}\, v$ = Dimensionless shot velocity

$\xi = \frac{x}{x_0}$ = Dimensionless shot travel

$\sigma = 1 - f$

3. Geometrical formulation of the ballistic problem

Denoting dx/dt and $1-f$ by v and σ respectively, the fundamental equations (1), (2), (3), (4), and (5) may be written as the autonomous system

$$\begin{aligned} \frac{dx}{dt} &= v, \\ \frac{dv}{dt} &= \frac{A}{M} P, \\ \frac{d\sigma}{dt} &= \frac{B}{W} P^{\alpha}, \end{aligned} \tag{6}$$

wherein

$$P = \frac{CF\sigma(1+\vartheta-\vartheta\sigma) - \frac{1}{2}(\gamma-1)Mv^2}{Ax}. \tag{7}$$

A solution of system (6) is interpreted as a space curve in (x, v, σ)-space with time t a parameter along the curve (see Fig. 1). The (x, v, σ)-space is called the *phase-space* of the system (6) and a solution is called a *trajectory*. The parameter t may be eliminated by dividing the first and second equations of (6) by the third.

Hence the trajectories are also solutions of the system,

$$\begin{aligned} \frac{dx}{d\sigma} &= \frac{W}{B}\,\frac{v}{P^{\alpha}}\,, \\ \frac{dv}{d\sigma} &= \frac{A}{M}\,\frac{W}{B}\,P^{1-\alpha}. \end{aligned} \tag{8}$$

We are interested in those solutions corresponding to a zero shot start pressure, *i.e.*, $P=0$ initially. The appropriate initial conditions are therefore

$$x = x_0 > 0, \qquad v = 0, \qquad \sigma = 0. \tag{9}$$

In the phase-space, these initial conditions are represented by a half-line, the positive x-axis. Since at each point of the half-line the right hand members in (6) vanish simultaneously, the initial conditions (9) correspond to critical points* of the system. Each of the latter constitutes a trivial solution (so called point trajectory) of (6) which is, of course, of no practical interest. However, equations (6) admit other non-trivial solutions emanating from the critical points. The latter therefore are not determined by their initial values alone. This lack of uniqueness is a result of the fractional nature of the exponent α. Indeed if $\alpha=1$, the system (6) is analytic at each point of the half-line (9), and only the trivial solution is admitted: This is an interesting point since a linear burning law, $\alpha=1$, is by far the most popular mathematical model, and the corresponding solutions are widely applied in practice. In § 6 we shall see that the latter violate the initial conditions with reference to the time variable although the geometrical conditions (9) in the phase-space are realized.

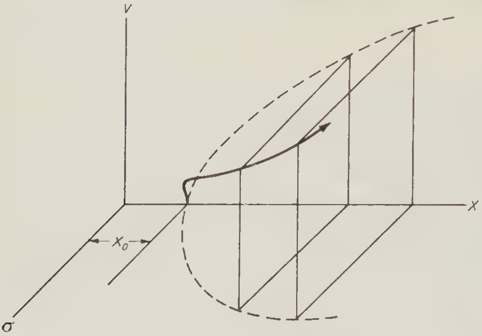

Fig. 1. Phase space trajectory

For $\alpha<1$, the second of equations (8) requires that

$$\frac{v}{\sigma} \to 0 \tag{10}$$

as the initial point $x=x_0$, $v=0$, $\sigma=0$ is approached along a trajectory. Thus (7) implies that

$$\frac{P}{\sigma} = \frac{CF(1+\vartheta-\vartheta\sigma) - \frac{\gamma-1}{2} M \frac{v^2}{\sigma}}{A\,x} \to \frac{CF(1+\vartheta)}{A\,x_0} \tag{11}$$

so that along a non-trivial solution, the pressure P vanishes as σ at the initial point**.

From the second of equations (8) we obtain

$$dv = \frac{1}{2-\alpha}\,\frac{A}{M}\,\frac{B}{W}\left(\frac{P}{\sigma}\right)^{1-\alpha} d\sigma^{2-\alpha},$$

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* See [6, Chapter II], for example.

** Generally $-1<\vartheta$, but for later considerations in § 9, we note here that if $\vartheta=-1$, then P vanishes as σ^2 and $\frac{P}{\sigma^2} \to \frac{CF}{A\,x_0}$.

and in view of (11), consider a change of independent variable to

$$\mu = \sigma^{2-\alpha}, \tag{12}$$

so that

$$\frac{dv}{d\mu} \to \frac{1}{2-\alpha}\,\frac{A}{M}\,\frac{W}{B}\left(\frac{CF(1+\vartheta)}{A\,x_0}\right)^{1-\alpha} \neq 0$$

at the initial point. Using (12), equations (8) become

$$\begin{aligned} \frac{dx}{d\mu} &= \frac{1}{2-\alpha}\,\frac{W}{B}\,\frac{v}{\mu^{\frac{1}{2-\alpha}}}\left(\frac{\mu^{\frac{1}{2-\alpha}}}{P}\right)^{\alpha}, \\ \frac{dv}{d\mu} &= \frac{1}{2-\alpha}\,\frac{A}{M}\,\frac{W}{B}\left(\frac{P}{\mu^{\frac{1}{2-\alpha}}}\right)^{1-\alpha}. \end{aligned} \tag{13}$$

At this point it is convenient to introduce the dimensionless variables

$$\begin{aligned} \nu &= \sqrt{\frac{M}{CF}}\,v, \\ \xi &= \frac{x}{x_0}. \end{aligned} \tag{14}$$

For conciseness, we also introduce the symbols

$$\begin{aligned} P_c &= \frac{CF}{A\,x_0}, \\ P_1 &= \left(\frac{B^2 M F C}{W^2 A^2}\right)^{\frac{1}{2-2\alpha}}. \end{aligned} \tag{15}$$

Each of the latter has the dimensions of a pressure. The first is, in fact, the closed chamber pressure, *i.e.*, the maximum pressure which would be produced upon burning the complete charge in the initial chamber volume. We shall see in § 5 that the quantity P_1 is closely associated with the maximum pressure of the actual ballistic process. If (14) and (15) are introduced into (13), there is obtained the system

$$\begin{aligned} \frac{d\xi}{d\mu} &= \beta\left(\frac{P_c}{P_1}\right)^{1-\alpha}\left(\frac{P_c\,\mu^\beta}{P}\right)^{\alpha}\frac{\nu}{\mu^\beta}, \\ \frac{d\nu}{d\mu} &= \beta\left(\frac{P_c}{P_1}\right)^{1-\alpha}\left(\frac{P}{P_c\,\mu^\beta}\right)^{1-\alpha}, \end{aligned} \tag{16}$$

wherein $\beta = \frac{1}{2-\alpha}$ and

$$\frac{P}{P_c\,\mu^\beta} = \frac{1+\vartheta-\vartheta\mu^\beta-\frac{\gamma-1}{2}\,\frac{\nu^2}{\mu^\beta}}{\xi}. \tag{17}$$

The initial conditions appropriate to equations (16) become

$$\begin{aligned} \xi &= 1, \\ \nu &= 0, \\ \frac{\nu}{\mu^\beta} &= 0, \\ \frac{P}{P_c\,\mu^\beta} &= 1+\vartheta \end{aligned} \tag{18}$$

for $\mu = 0$. The latter two follow from (10) and (11) using (12). Owing to the condition $P/P_c \mu^\beta = 1 + \vartheta \neq 0$, the initial point of the trajectory in (ξ, ν, μ)-space no longer appears as a critical point. Thus (16) exhibit only the non-trivial solutions. The system remains non-analytic at the initial point, so that existence and uniqueness questions remain. However, the latter are settled by standard results from differential equations theory*: There exists a unique solution of (16) satisfying the initial conditions (18). Such a solution is the non-trivial solution of the original boundary value problem (6), (9). The formulation (16) is therefore fundamental to the interior ballistics problem.

4. General properties of the trajectories

If a solution of (16) is interpreted as a space curve or trajectory in (ξ, ν, μ)-space, then the trajectory is limited to that portion of the first octant for which $\xi \geqq 1$. In addition, since the solution exists only for $\frac{P}{P_c} \geqq 0$, equation (17) implies that the trajectory must also lie within the surface S defined by

$$(1+\vartheta)\,\mu^\beta - \vartheta\,\mu^{2\beta} - \frac{\gamma-1}{2}\nu^2 = 0. \quad (19)$$

For $\vartheta > 0$, S is an elliptic-shaped cylinder, bounded in the μ and ν directions (Fig. 2), while for $\vartheta \leqq 0$, S is a hyperbolic-shaped cylinder, open in the μ and ν directions (Fig. 3). In addition, a physically significant trajectory is limited to the halfspace $\mu \leqq 1$, since the charge C is completely burnt at $\mu = 1$. However, as this limitation constitutes an artificial discontinuity in the system (16) which bears no mathematical significance, we shall not impose it generally**.

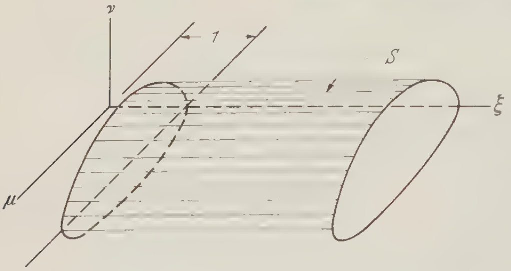

Fig. 2. Surface, S, for $0 < \vartheta$

Figs. 2 and 3 suggest that the complete mathematical trajectories for $\vartheta > 0$ may differ considerably from those for $\vartheta \leqq 0$. That this is indeed the case we shall see in § 5. However, some observations of a general nature may be made here. All trajectories emanate from the same point, $\xi = 1$, $\mu = \nu = 0$, with a tangent in the $\xi = 1$ plane of finite slope. This is in contrast to the vertical (ν-oriented) tangent at the initial points of the trajectories in the phase-space.

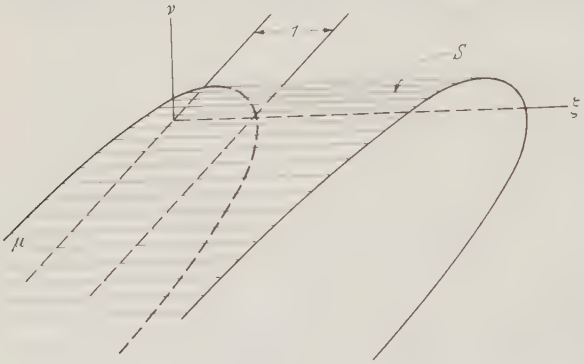

Fig. 3. Surface, S, for $\vartheta \leqq 0$

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* See [7, Chapter I], for example.

** A discussion of the complete space curve solution is both instructive and, as we shall see in §§ 7, 8, and 10, of practical concern.

Since $\frac{d\nu}{d\mu} \geqq 0$ throughout, a trajectory continuously rises in the ν-direction and twists toward ξ increasing. Such a trajectory will thus terminate only as it contacts the surface S, at which point the pressure P will vanish. For $\vartheta > 0$, it is possible for a trajectory to avoid the surface S only if it progresses up (ξ increasing) the elliptic-shaped cylinder. However, in such a case since $\xi \rightarrow \infty$ and both μ and ν approach finite limits, (17) implies that P will vanish. Thus in any case for $\vartheta > 0$, P must ultimately return to zero for some finite value of μ and the mathematical solution terminates. We note in passing that P necessarily attains a mathematical maximum.

In contrast, we shall show in § 5 that for $\vartheta \leqq 0$, the pressure cannot return to zero so that the mathematical solution may be continued for all μ. In addition, P will approach zero as $\mu \rightarrow \infty$ only for $\vartheta = 0$. Hence, a mathematical maximum for P when $\vartheta < 0$ appears uncertain. It is well known, for example, that some solutions for the linear burning rate, $\alpha = 1$, exhibit continuously increasing pressure characteristics. In §§ 5 and 6 it will be shown that $\alpha = 1$ is, however, exceptional in this respect.

5. Pressure characteristics

We rewrite (17) as

$$\xi \frac{P}{P_c} = (1+\vartheta)\,\mu^{\beta} - \vartheta\,\mu^{2\beta} - \frac{\gamma-1}{2}\,\nu^2 \tag{20}$$

and compute the ν-derivative of (20). Using the second of equations (16), we obtain

$$\frac{d}{d\nu}\left(\xi \frac{P}{P_c}\right) = (1+\vartheta-2\vartheta\,\mu^{\beta})\left(\frac{P_1}{P}\right)^{1-\alpha} - (\gamma-1)\,\nu\,.$$

On the other hand we have

$$\frac{d}{d\nu}\left(\xi \frac{P}{P_c}\right) = \xi \frac{d}{d\nu}\left(\frac{P}{P_c}\right) + \frac{P}{P_c}\frac{d\xi}{d\nu}\,,$$

and, upon dividing the first of equations (16) by the second, we obtain

$$\frac{d\xi}{d\nu} = \frac{P_c}{P}\,\nu\,. \tag{21}$$

Hence,

$$\xi \frac{d}{d\nu}\left(\frac{P}{P_c}\right) = \mu^{\beta}\left(\frac{P_1}{P}\right)^{1-\alpha}\left[\frac{1+\vartheta}{\mu^{\beta}} - 2\vartheta - \gamma\left(\frac{P}{P_1}\right)^{1-\alpha}\frac{\nu}{\mu^{\beta}}\right]. \tag{22}$$

Considering once again the second of equations (16) expressed in the form

$$d\nu = \left(\frac{P}{P_1}\right)^{1-\alpha} d\mu^{\beta}, \tag{23}$$

we may write

$$\nu = \int_0^{\mu^{\beta}} \left(\frac{P}{P_1}\right)^{1-\alpha} d\mu^{\beta}.$$

Hence the quantity ν/μ^{β} is the average value of $(P/P_1)^{1-\alpha}$ as a function of μ^{β}. It is convenient here to introduce a function P_a which depicts an average value of P itself. Accordingly we define

$$\frac{P_a}{P_1} = \left(\frac{\nu}{\mu^\beta}\right)^{\frac{1}{1-\alpha}} = \left[\frac{1}{\mu^\beta}\int\limits_0^{\mu^\beta}\left(\frac{P}{P_1}\right)^{1-\alpha} d\mu^\beta\right]^{\frac{1}{1-\alpha}}. \tag{24}$$

Thus (22) may be written as

$$\xi\frac{d}{d\nu}\left(\frac{P}{P_c}\right) = \mu^\beta\left(\frac{P_1}{P}\right)^{1-\alpha}\left[\frac{1+\vartheta}{\mu^\beta} - 2\vartheta - \gamma\left(\frac{P}{P_1}\right)^{1-\alpha}\left(\frac{P_a}{P_1}\right)^{1-\alpha}\right]. \tag{25}$$

We now observe that P cannot increase without bound, for then the third term in the brackets of (25) would dominate the other two and $d/d\nu\,(P/P_c)$ would ultimately become negative. Thus the pressure, P, must exhibit a mathematical maximum. This was shown in § 4 for $\vartheta > 0$ but now applies generally. For $\vartheta < 0$, (25) ostensibly permits P to increase monotonically to its maximum (finite) value as $\mu \to \infty$. However, the ν-derivative of (25) reveals that in such a case $d^2(P/P_c)/d\nu^2$ would ultimately become and remain positive, which in turn would lead to an unbounded P/P_c. Hence, in every case a maximum* of P, which we denote by P_m, occurs for finite μ. The value of the latter we denote by μ_m.

Since $(dP/P_c)/d\nu$ vanishes for $P = P_m$ and certainly $P_a \leqq P_m$, (25) requires that

$$\frac{1+\vartheta}{\mu_m^\beta} - 2\vartheta \leqq \gamma\left(\frac{P_m}{P_1}\right)^{2-2\alpha}, \tag{26}$$

and *a fortiori*

$$-\frac{2\vartheta}{\gamma} \leqq \left(\frac{P_m}{P_1}\right)^{2-2\alpha}.$$

For $\vartheta > 0$, the latter is trivial but for $\vartheta < 0$ yields the important inequality

$$\left[-\frac{2\vartheta B^2 M F C}{\gamma W^2 A^2}\right]^{\frac{1}{2-2\alpha}} \leqq P_m. \tag{27}$$

An upper bound to P_m is also available. In fact, using (24) we may rewrite (17) in the form

$$\frac{P}{P_c\,\mu^{2\beta}} = \frac{\dfrac{1+\vartheta}{\mu^\beta} - \vartheta - \dfrac{\gamma - 1}{2}\left(\dfrac{P_a}{P_1}\right)^{2-2\alpha}}{\xi}.$$

Thus $P/P_c\,\mu^{2\beta}$ decreases monotonically up to maximum P. Rewriting (23) in the form

$$d\nu = \frac{\beta}{2-\beta}\left(\frac{P}{P_1\,\mu^{2\beta}}\right)^{1-\alpha} d\mu^{2-\beta},$$

we conclude that

$$\nu \geqq r\left(\frac{P}{P_1\,\mu^{2\beta}}\right)^{1-\alpha}\mu^{2-\beta}$$

or

$$\frac{\nu}{\mu^\beta} \geqq r\left(\frac{P}{P_1}\right)^{1-\alpha}$$

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* Strictly speaking, we refer here and throughout to the first mathematical maximum.

up to maximum P. Hence from (22) we have

$$\frac{1+\vartheta}{\mu_m^\beta}-2\vartheta \geqq r\gamma\left(\frac{P_m}{P_1}\right)^{2-2\alpha}, \tag{28}$$

and (26) and (28) together yield

$$\left[\frac{\frac{1+\vartheta}{\mu_m^\beta}-2\vartheta}{\gamma}\right]^{\frac{1}{2-2\alpha}} \leqq \frac{P_m}{P_1} \leqq (3-2\alpha)^{\frac{1}{2-2\alpha}}\left[\frac{\frac{1+\vartheta}{\mu_m^\beta}-2\vartheta}{\gamma}\right]^{\frac{1}{2-2\alpha}}. \tag{29}$$

It might be noted that $3^{\frac{1}{2}} \leqq (3-2\alpha)^{\frac{1}{2-2\alpha}} \leqq e$ for $0 \leqq \alpha \leqq 1$. Actually the bounds in (29) are not determined since the value of μ_m^β is in general unknown. However, for the special case $\vartheta = -1$ considered in § 9, (29) becomes

$$\left(\frac{2}{\gamma}\right)^{\frac{1}{2-2\alpha}} \leqq \frac{P_m}{P_1} \leqq (3-2\alpha)^{\frac{1}{2-2\alpha}}\left(\frac{2}{\gamma}\right)^{\frac{1}{2-2\alpha}}. \tag{30}$$

It should also be observed that as the maximum pressure is delayed, *i.e.*, μ_m^β becomes large, (29) approaches the explicit form (30). Another observation of interest here is that according to (25) the unknown quantity in (29) is given by

$$\frac{\frac{1+\vartheta}{\mu_m^\beta}-2\vartheta}{\gamma} = \left(\frac{P_m}{P_1}\right)^{1-\alpha}\left(\frac{P_a}{P_1}\right)^{1-\alpha}. \tag{31}$$

Using (31) in (29) results in

$$P_a \leqq P_m \leqq (3-2\alpha)^{\frac{1}{1-\alpha}} P_a, \tag{32}$$

which relates to the piezometric efficiency. For convenience here we define* the latter by the ratio $\varepsilon_p = \frac{P_a}{P_m}$, so that (32) yields $\frac{1}{(3-2\alpha)^{\frac{1}{1-\alpha}}} \leqq \varepsilon_p \leqq 1$.

In § 4 we observed that for $\vartheta > 0$, the pressure P returns to zero for finite μ. However, if $\vartheta < 0$, from (25) we see that P cannot decrease to zero since $d(P/P_c)/dv$ would then become positive**.

Thus for $\vartheta \leqq 0$ it is of some interest to consider the possible limiting values of P as $\mu \to \infty$. It is not difficult to obtain an asymptotic solution of equations (16). The latter is a *constant* pressure solution with

$$P = \left(-\frac{2\vartheta}{\gamma}\right)^{\frac{1}{2-2\alpha}} \cdot P_1 = \left(\frac{-2\vartheta B^2 M F C}{\gamma W^2 A^2}\right)^{\frac{1}{2-2\alpha}}. \tag{33}$$

It is interesting to note that (33) is given by the lower bound to P_m obtained previously, (27).

We infer then that for $\vartheta \leqq 0$, the pressure curve rises to a maximum which exceeds the value in (33) and subsequently decays and approaches the asymptotic solution (33). In § 9 we shall again consider the latter together with its special significance for ideally progressive burning propellants.

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* Piezometric efficiency is generally referred to muzzle values only and depicts a displacement-average pressure to maximum pressure ratio.

** For $\vartheta = 0$, this argument fails with $\mu \to \infty$ since P may tend to zero with the remaining term $\frac{1+\vartheta}{\mu^\beta}$.

6. The linear burning law, $\alpha = 1$

It is well known that ballistic solutions can be obtained in closed form for $\alpha = 1$. Indeed much of ballistic technology is based on these solutions despite overwhelming experimental evidence suggesting that the appropriate values of α should be less than unity. It is important therefore to understand in what sense the $\alpha = 1$ solutions and the $\alpha < 1$ solutions resemble one another. Not so well known are many significant differences: Actually, the linear burning case is exceptional in many respects as it exhibits numerous phenomena not exhibited by any other case. The results in § 5 indicate some of these variances. For example, from (27), it is to be expected that when the quantity $\frac{-2\vartheta}{\gamma} \cdot \frac{B^2 M F C}{W^2 A^2}$ exceeds unity, the maximum pressure, P_m, increases with increasing α. In fact, for $\alpha \to 1$, (27) requires that $P_m \to \infty$. Hence, the $\alpha = 1$ solutions would not be expected to exhibit maximum pressures when $\frac{-2\vartheta}{\gamma} \cdot \frac{B^2 M F C}{W^2 A^2} > 1$. The explicit solutions, which we shall consider presently, verify this conclusion. However, the latter is in sharp contrast to the fact that for $\alpha < 1$, every solution exhibits a maximum pressure.

To consider the limit $\alpha \to 1$, we rewrite equations (16) in the form

$$\frac{d\xi}{d\mu} = \beta \frac{P_c^{1-\alpha}}{P_2} \left(\frac{P_c \mu^\beta}{P}\right)^\alpha \frac{\nu}{\mu^\beta}, \qquad \frac{d\nu}{d\mu} = \beta \frac{P_c^{1-\alpha}}{P_2} \left(\frac{P}{P_c \mu^\beta}\right)^{1-\alpha}, \tag{34}$$

wherein $P_2 = P_1^{1-\alpha} = \left(\frac{B^2 M F C}{W^2 A^2}\right)^{\frac{1}{2}}$ is independent of α. Since $\frac{P}{P_c \mu^\beta}$ is bounded away from zero initially, the solutions depend continuously upon the parameter α. In view of the above remarks concerning maximum pressures, we conclude that when the quantity $\frac{-2\vartheta}{\gamma} \cdot \frac{B^2 M F C}{W^2 A^2} = \frac{-2\vartheta}{\gamma} \cdot P_2^2$ exceeds unity, the maximum pressures for $\alpha > 1$ must occur for increasingly larger values of μ as $\alpha \to 1$ (*i.e.*, $\mu_m \to \infty$).

With $\alpha = 1$, the solution of (34) is*

$$\begin{aligned} \xi &= \left[\frac{2(1+\vartheta) P_2}{2(1+\vartheta) P_2 - (2\vartheta P_2^2 + \gamma - 1)\nu}\right]^{\frac{2}{2\vartheta P_2^2 + \gamma - 1}}, \\ \nu &= \frac{1}{P_2} \mu . \end{aligned} \tag{35}$$

The corresponding pressure ratio is

$$\frac{P}{P_c} = \frac{1}{2} \left[2 P_2 (1+\vartheta)\nu - (2 P_2^2 \vartheta + \gamma - 1)\nu^2\right] \left[\frac{2 P_2 (1+\vartheta) - (2\vartheta P_2^2 + \gamma - 1)\nu}{2 P_2 (1+\vartheta)}\right]^{\frac{2}{2\vartheta P_2^2 + \gamma - 1}}, \tag{36}$$

so that

$$\frac{d(P/P_c)}{d\nu} = \left[P_2 (1+\vartheta) - (2\vartheta P_2^2 + \gamma)\nu\right] \left[\frac{2 P_2 (1+\vartheta) - (2\vartheta P_2^2 + \gamma - 1)\nu}{2 P_2 (1+\vartheta)}\right]^{\frac{2}{(2\vartheta P_2^2 + \gamma - 1)}}. \tag{37}$$

---

* We assume here that $2\vartheta P_2^2 + \gamma - 1 \neq 0$. If $2\vartheta P_2^2 + \gamma - 1 = 0$, then $\xi = \exp \cdot \frac{\nu}{(1+\vartheta) P_2}$.

Hence a maximum of P/P_c occurs for*

$$\nu = \frac{(1+\vartheta)\,P_2}{2\vartheta P_2^2+\gamma}, \tag{38}$$

if $0 < 2\vartheta P_2^2 + \gamma$, *i.e.*, if $\frac{-2\vartheta}{\gamma} P_2^2 < 1$.

The maximum value of P/P_c becomes

$$\frac{P_m}{P_c} = \frac{P_2^2\,(1+\vartheta)^2}{2\vartheta P_2^2+\gamma}\left[\frac{2\vartheta P_2^2+\gamma+1}{2(2\vartheta P_2^2+\gamma)}\right]^{\frac{2\vartheta P_2^2+\gamma+1}{2\vartheta P_2^2+\gamma-1}},$$

and, if $2\vartheta P_2^2+\gamma-1>0$, $\frac{P}{P_c}$ subsequently returns to zero for $\nu = \frac{2(1+\vartheta)\,P_2}{2\vartheta P_2^2+\gamma-1}$ while $\xi\to\infty$. However if $2\vartheta P_2^2+\gamma-1<0$**, $\frac{P}{P_c}$ returns to zero only as $\nu\to\infty$. For $2\vartheta P_2^2+\gamma<0$, *i.e.*, $\frac{-2\vartheta P_2^2}{\gamma} > 1$, equation (37) implies that $\frac{P}{P_c}$ increases monotonically *without bound* as $\nu\to\infty$ and $\xi\to\infty$.

These results give strong indications that parameter effects on various ballistic quantities such as maximum pressure with $\alpha<1$ may not be adequately reflected in the $\alpha=1$ solutions. Some considerations of these effects appear in §§ 7 and 8. Further evidence of the peculiar nature of the $\alpha=1$ solutions is afforded by considerations of the corresponding phase-space trajectories. The latter are defined by the equations

$$\begin{aligned} x &= x_0\,\xi, \\ v &= \sqrt{\frac{\bar C\,\bar F}{M}}\,\nu, \\ \sigma &= \mu, \end{aligned} \tag{39}$$

where ξ and ν are given by equations (35). Although $x=x_0$, $v=0$, $\sigma=0$ is the initial point of the trajectories for $\alpha<1$, it is attained in the limit only for $t\to-\infty$ along the limiting trajectory (39)***. Indeed, equations (34) with $\alpha=1$ imply that

$$\frac{d\xi}{\nu} = \frac{P_c}{P_2}\,\frac{d\mu}{P} = P_c\,\frac{d\nu}{P} \tag{40}$$

is not integrable at $\nu=0$. However $d\xi/\nu$ is proportional to the differential of time dt. Hence the parameter t cannot be introduced**** along the trajectory (39) so as to yield a solution of equations (6) satisfying the initial conditions (9).

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* Note that if $2\vartheta P_2^2+\gamma-1>0$, then ν is bounded by $\frac{2(1+\vartheta)\,P_2}{2\vartheta P_2^2+\gamma-1}$, which, however, exceeds the value in (38), so that the maximum is actually attained. If $2\vartheta P_2^2+\gamma-1<0$, then ν is not bounded.

** This inequality together with the condition $2\vartheta P_2^2+\gamma>0$ requires $0<2\vartheta P^2+\gamma<1$.

*** The solution (39) might be interpreted as a 3-dimensional *separatrix* [6, Chapter V].

**** A device such as used by Corner [5, Chapter 4] to avoid the singularity in (40) at $\nu=0$ merely truncates the solution so that both ν and P fail to vanish initially.

7. The geometric parameter, ϑ

In this section and in §§ 8 and 9 we consider similarity transformations which form the basis for parameter studies*. These transformations are not original here, although the present treatment and use is somewhat novel.

The importance of the geometric parameter ϑ has been noted previously in the general discussion of the solutions. In particular, it was observed that the two ranges $\vartheta \leqq 0$ and $\vartheta > 0$ exhibit marked differences with reference to the trajectories and the pressure curves. Still, the solutions of equations (16) depend continuously** upon ϑ in the range $-1 < \vartheta \leqq 1$***, so that some similarities must also be present. The latter may be exhibited through the similarity transformations.

We note that for $\vartheta \neq -1$, equation (17) may be written

$$\frac{2}{1+\vartheta}\,\frac{P}{P_c\,\mu^\beta} = \frac{2 - \frac{2\vartheta}{1+\vartheta}\,\mu^\beta - \frac{\gamma-1}{1+\vartheta}\,\frac{v^2}{\mu^\beta}}{\xi}. \tag{41}$$

For $\vartheta = 1$, (41) becomes

$$\frac{P}{P_c\,\mu^\beta} = \frac{2 - \mu^\beta - \frac{\gamma-1}{2}\,\frac{v^2}{\mu^\beta}}{\xi}. \tag{42}$$

Equations (41) and (42) suggest the change of variables, say for $\vartheta > 0$, to

$$\begin{aligned} \mu_1 &= \left(\frac{2\vartheta}{1+\vartheta}\right)^{\frac{1}{\beta}} \mu\,, \\ v_1 &= \frac{2\vartheta^{\frac{1}{2}}}{1+\vartheta}\,v\,. \end{aligned} \tag{43}$$

For then (41) becomes

$$\frac{4\vartheta}{(1+\vartheta)^2}\,\frac{P}{P_c\,\mu_1^\beta} = \frac{2 - \mu_1^\beta - \frac{\gamma-1}{2}\,\frac{v_1^2}{\mu_1^\beta}}{\xi}, \tag{44}$$

which is (42) except for notation. It should be observed that equations (43) merely effect scalar changes in variables which are subjected to homogeneous initial conditions. Introducing the new variables (43) into equations (16) and using (44), we obtain

$$\begin{aligned} \frac{d\xi}{d\mu_1} &= \beta\left[\frac{(1+\vartheta)^2}{4\vartheta^{\left(\frac{3-2\alpha}{2-2\alpha}\right)}}\,\frac{P_c}{P_1}\right]^{1-\alpha} \left[\frac{\xi}{2 - \mu_1^\beta - \frac{\gamma-1}{2}\,\frac{v_1^2}{\mu_1^\beta}}\right]^{\alpha} \frac{v_1}{\mu_1^\beta}, \\ \frac{dv_1}{d\mu_1} &= \beta\left[\frac{(1+\vartheta)^2}{4\vartheta^{\left(\frac{3-2\alpha}{2-2\alpha}\right)}}\,\frac{P_c}{P_1}\right]^{1-\alpha} \left[\frac{2 - \mu_1^\beta - \frac{\gamma-1}{2}\,\frac{v_1^2}{\mu_1^\beta}}{\xi}\right]^{1-\alpha}, \end{aligned} \tag{45}$$

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* RÖGGLA [*4*] and CORNER [*5*] have employed similarity transformations extensively as a means of organizing tabular solutions, *i.e.*, ballistic tables.

** This statement must be qualified, of course, with reference to the extent of the solutions.

*** $\vartheta = -1$ is exceptional and is considered in § 9.

which are equations (16) in the new variables (ξ, μ_1, ν_1) with $\vartheta = 1$ and P_c/P_1 replaced by $\dfrac{(1+\vartheta)^2}{4\vartheta^{\left(\frac{3-2\alpha}{2-2\alpha}\right)}} \dfrac{P_c}{P_1}$. Hence any solution with $0 < \vartheta < 1$ is given by a corresponding $\vartheta = 1$ solution. We note here that the pressure ratio P/P_c in (16) is replaced by $\dfrac{4\vartheta}{(1+\vartheta)^2} \dfrac{P}{P_c}$ in (45). Since ξ is invariant, it is convenient to regard ξ as the independent variable throughout. Equations (43) are then interpreted as functional equations holding identically in ξ $(\xi \geqq 1)$, and all properties of the solutions for $0 < \vartheta < 1$ are reflected in the $\vartheta = 1$ solutions*. In particular, the pressure ratio is given by

$$\frac{P}{P_c} = \frac{(1+\vartheta)^2}{4\vartheta} \left(\frac{P}{P_c}\right)_{\vartheta=1}, \tag{46}$$

holding identically in ξ, wherein $(P/P_c)_{\vartheta=1}$ is the pressure ratio given by (16) for $\vartheta = 1$ and for an equivalent ballistic parameter

$$\left(\frac{P_c}{P_1}\right)_e = \frac{(1+\vartheta)^2}{4\vartheta^{\left(\frac{3-2\alpha}{2-2\alpha}\right)}} \frac{P_c}{P_1}. \tag{47}$$

The effects of variations of ϑ on the pressure ratio are therefore two-fold: A simple scalar effect as given by (46) and the more subtle effect inferred from ϑ variations on the equivalent ballistic parameter, (47). The similarity transformation (43) limits ϑ to the range $0 < \vartheta \leqq 1$. This, of course, is to be expected from the discussions in §§ 4 and 5, since the solutions for $0 < \vartheta \leqq 1$ terminate with finite μ (*i.e.*, finite μ_1), while for $-1 < \vartheta \leqq 0$, they are defined for all μ.

For ϑ in the range $-1 < \vartheta < 0$, similarity transformations are also available. By way of illustration we may choose $\vartheta = -\frac{1}{2}$ as a standard. The transformation to other ϑ in $-1 < \vartheta < 0$ is then defined by the equations

$$\begin{aligned} \mu_2 &= \left(\frac{-\vartheta}{1+\vartheta}\right)^{\frac{1}{\beta}} \mu, \\ \nu_2 &= \frac{\left(-\frac{\vartheta}{2}\right)^{\frac{1}{2}}}{1+\vartheta} \nu, \end{aligned} \tag{48}$$

wherein the triplet (ξ, ν_2, μ_2) is the trajectory given by equations (16) for $\vartheta = -\frac{1}{2}$, and for the equivalent ballistic parameter

$$\left(\frac{P_c}{P_1}\right)_e = \frac{(1+\vartheta)^2}{2^{\left(\frac{1-2\alpha}{2-2\alpha}\right)} (-\vartheta)^{\left(\frac{3-2\alpha}{2-2\alpha}\right)}} \frac{P_c}{P_1}. \tag{49}$$

The pressure curve is transformed according to

$$\frac{P}{P_c} = \frac{2(1+\vartheta)^2}{(-\vartheta)} \left(\frac{P}{P_c}\right)_{\vartheta=-\frac{1}{2}}. \tag{50}$$

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* It might be noted, however, that "burnt" determined by $\mu = 1$ now becomes $\mu_1 = \left(\dfrac{2\vartheta}{1+\vartheta}\right)^{\frac{1}{\beta}}$.

In this instance, however, it should be noted that "burnt" is now given by $\mu_2 = \left(\frac{-\vartheta}{1+\vartheta}\right)^{\frac{1}{\beta}}$, which varies between 0 and $+\infty$ with ϑ in $-1 < \vartheta < 0$. Hence, the complete mathematical trajectory is required if the similarity transformation (48) is to be used in general. Again ξ is invariant, and all properties of the solutions for $-1 < \vartheta < 0$ are reflected in the $\vartheta = -\frac{1}{2}$ solutions.

The similarity transformations (43) and (48) amply demonstrate the continuous dependence of the solutions upon ϑ in the two ranges $-1 < \vartheta < 0$ and $0 < \vartheta < 1$. The two limiting cases $\vartheta \to 0$ and $\vartheta \to -1$ are not covered, however. These are treated (although rather incidentally) in the following two sections.

8. Ballistic parameters

We consider a general similarity transformation

$$\begin{aligned} \xi &= \xi, \\ \mu_3 &= \left(\frac{P_c}{P_1}\right)^a \mu, \\ \nu_3 &= \left(\frac{P_c}{P_1}\right)^b \nu, \end{aligned} \tag{51}$$

wherein a and b are constants. In terms of the new variables, equations (16) become

$$\begin{aligned} \frac{d\xi}{d\mu_3} &= \beta \left(\frac{P_c}{P_1}\right)^{1-\alpha-\beta(1-\alpha)a-b} \times \\ &\times \left[\frac{\xi}{(1+\vartheta)\,\mu_3^\beta - \left(\frac{P_c}{P_1}\right)^{-\beta a} \vartheta\,\mu_3^{2\beta} - \frac{\gamma-1}{2}\left(\frac{P_c}{P_1}\right)^{-2b+\beta a} \nu_3^2}\right]^\alpha \frac{\nu_3}{\mu_3^{1-\beta}}, \\ \frac{d\nu_3}{d\mu_3} &= \beta \left(\frac{P_c}{P_1}\right)^{1-\alpha-\beta(2-\alpha)a+b} \times \\ &\times \left[\frac{(1+\vartheta)\,\mu_3^\beta - \left(\frac{P_c}{P_1}\right)^{-\beta a} \vartheta\,\mu_3^{2\beta} - \frac{\gamma-1}{2}\left(\frac{P_c}{P_1}\right)^{-2b+\beta a} \nu_3^2}{\xi}\right]^{1-\alpha} \frac{1}{\mu_3^{1-\beta}}. \end{aligned} \tag{52}$$

We should like to choose a and b so as to eliminate any effects of variations of the ballistic parameter P_c/P_1 in (52). This is not possible in general, but if we choose a and b so that

$$1 - \alpha - \beta(1-\alpha)\,a - b = 0,$$

$$1 - \alpha - \beta(2-\alpha)\,a + b = 0$$

$\left(i.e.,\ a = \left(\frac{2-2\alpha}{3-2\alpha}\right)\frac{1}{\beta},\ b = \frac{1-\alpha}{3-2\alpha}\right)$, then equations (52) become*

$$\begin{aligned} \frac{d\xi}{d\mu_3} &= \beta \left[\frac{\xi}{1+\vartheta - \left(\frac{P_1}{P_c}\right)^{\left(\frac{2-2\alpha}{3-2\alpha}\right)} \vartheta\,\mu_3^\beta - \frac{\gamma-1}{2}\,\frac{\nu_3^2}{\mu_3^\beta}}\right]^\alpha \frac{\nu_3}{\mu_3^\beta}, \\ \frac{d\nu_3}{d\mu_3} &= \beta \left[\frac{1+\vartheta - \left(\frac{P_1}{P_c}\right)^{\left(\frac{2-2\alpha}{3-2\alpha}\right)} \vartheta\,\mu_3^\beta - \frac{\gamma-1}{2}\,\frac{\nu_3^2}{\mu_3^\beta}}{\xi}\right]^{1-\alpha}, \end{aligned} \tag{53}$$

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* Note that rather fortuitiously $-2b + \beta a = 0$.

and (51) becomes

$$\begin{aligned} \mu_3 &= \left(\frac{P_c}{P_1}\right)^{\left(\frac{2-2\alpha}{3-2\alpha}\right)\frac{1}{\beta}} \mu\,, \\ \nu_3 &= \left(\frac{P_c}{P_1}\right)^{\left(\frac{1-\alpha}{3-2\alpha}\right)} \nu\,, \end{aligned} \tag{54}$$

while the pressure ratio is given by

$$\frac{P}{P_c}\left(\frac{P_c}{P_1}\right)^{\left(\frac{2-2\alpha}{3-2\alpha}\right)} = \frac{(1+\vartheta)\,\mu_3^{\beta} - \vartheta\left(\frac{P_1}{P_c}\right)^{\left(\frac{2-2\alpha}{3-2\alpha}\right)} \mu_3^{2\beta} - \frac{\gamma-1}{2}\,\nu_3^2}{\xi}\,. \tag{55}$$

Hence any effects of variations of the ballistic parameter are now reflected either in the coefficient of the $[-\vartheta\mu_3^{\beta}]$ terms in (53) or the simple scalar effects in (54) and (55)★.

For $\vartheta = 0$, the $[-\vartheta\mu_3^{\beta}]$ terms are not present, and variations of the ballistic parameter can effect only scalar changes as in (54) and (55)★★. In particular, the maximum pressure is then proportional to

$$P_3 = P_c\left(\frac{P_1}{P_c}\right)^{\left(\frac{2-2\alpha}{3-2\alpha}\right)}. \tag{56}$$

Using (15), we may express P_3 in physically significant quantities as

$$P_3 = \left[\frac{B^2 M F^2 C^2}{W^2 A^2 x_0}\right]^{\frac{1}{3-2\alpha}}, \tag{57}$$

which is the Clemmow ballistic parameter. Note that (56) may be written in the form

$$\frac{P_3}{P_c} = \left(\frac{P_1}{P_c}\right)^{\left(\frac{2-2\alpha}{3-2\alpha}\right)}. \tag{58}$$

By means of (58) one may interpret parameter effects with reference to either ballistic parameter. For example, the equivalent $\vartheta = 1$ ballistic parameter value in (47) becomes★★★

$$\left(\frac{P_3}{P_c}\right)_e = \frac{4^{\left(\frac{2-2\alpha}{3-2\alpha}\right)}\,\vartheta}{(1+\vartheta)^{\left(\frac{4-4\alpha}{3-2\alpha}\right)}}\left(\frac{P_3}{P_c}\right), \tag{59}$$

and the pressure ratio is computed according to

$$\frac{P}{P_3} = \left(\frac{1+\vartheta}{2}\right)^{\frac{2}{3-2\alpha}}\left(\frac{P}{P_3}\right)_{\vartheta=1}. \tag{60}$$

It is interesting to observe that with the present formulation [*i.e.*, equations (53)] of the ballistic problem, we are now able to obtain the $\vartheta = 0$ solution from a standard $\vartheta = 1$ solution. In fact according to (59) the former is given by the

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★ Note that "burnt" now becomes $\mu_3 = \left(\frac{P_c}{P_1}\right)^{\left(\frac{2-2\alpha}{3-2\alpha}\right)\frac{1}{\beta}}$.

★★ This is a well known result due to CLEMMOW [2]. Indeed, it is CLEMMOW's result which suggests the particular choices of the constants a and b.

★★★ Here we invert the parameter ratio for convenience in the subsequent discussions.

$\vartheta = 1$ solution with the equivalent ballistic parameter $(P_3/P_c)_e = 0$. According to (60) the corresponding pressure ratio is given by

$$\frac{P}{P_3} = \frac{1}{2^{\left(\frac{2}{3-2\alpha}\right)}} \left(\frac{P}{P_3}\right)_{\vartheta=1}. \tag{61}$$

The continuous dependence of the solution on the geometric parameter ϑ in the range $0 \leqq \vartheta \leqq 1$ (including zero) is thus illustrated. It is not difficult to show that the $\vartheta = 0$ solution may also be obtained from the standard $\vartheta = -\frac{1}{2}$ solution with the equivalent ballistic parameter value $(P_3/P_c)_e = 0$. Thus *every* solution with $P_3/P_c = 0$, regardless of the value of $\vartheta\,(-1 < \vartheta)$ is, except for notation, the $\vartheta = 0$ solution.

9. Ideally progressive geometry, $\vartheta = -1$

We are concerned here with the exceptional case $\vartheta = -1$. Equations (16) and (17) may be written

$$\begin{aligned} \frac{d\xi}{d\mu} &= \beta \left(\frac{P_c}{P_1}\right)^{1-\alpha} \left[\frac{P_c\,\mu^{2\beta}}{P}\right]^{\alpha} \frac{\nu}{\mu^{\beta(1+\alpha)}}, \\ \frac{d\nu}{d\mu} &= \beta \left(\frac{P_c}{P_1}\right)^{1-\alpha} \left[\frac{P}{P_c\,\mu^{2\beta}}\right]^{1-\alpha} \mu^{1-\beta} \end{aligned} \tag{62}$$

with

$$\frac{P}{P_c\,\mu^{2\beta}} = \frac{1 - \frac{\gamma-1}{2}\left(\frac{\nu}{\mu^{\beta}}\right)^2}{\xi}. \tag{63}$$

The appropriate initial conditions become

$$\begin{aligned} \xi &= 1, \\ \nu &= 0, \\ \frac{\nu}{\mu^{\beta}} &= 0, \\ \frac{P}{P_c\,\mu^{2\beta}} &= 1 \end{aligned} \tag{64}$$

for $\mu = 0$. We consider a general similarity transformation

$$\begin{aligned} \mu_4 &= \left(\frac{P_c}{P_1}\right)^{n} \mu^{2-\beta}, \\ \nu_4 &= \left(\frac{P_c}{P_1}\right)^{m} \nu, \end{aligned} \tag{65}$$

where m and n are constants. The exponent $2-\beta$ in the first of equations (65) is chosen so as to obtain a non-critical version of equations (62)*. By use of (65), equation (63) may be written:

$$\left(\frac{P_c}{P_1}\right)^{2nr} \frac{P}{P_1\,\mu_4^{2r}} = \frac{1 - \frac{\gamma-1}{2}\left(\frac{P_1}{P_c}\right)^{2(m-nr)} \left(\frac{\nu_4}{\mu_4^{r}}\right)^2}{\xi}. \tag{66}$$

---

* $\frac{d\nu}{d\mu^{2-\beta}} \to \frac{\beta}{2-\beta}\left(\frac{P_c}{P_1}\right)^{1-\alpha} \neq 0$ initially.

It is thus convenient to require that $m=n r$. Then with $n=\frac{3-2\alpha}{2}=\frac{1}{2r}$ (*i.e.*, $m=\frac{1}{2}$), equations (62) and (63) become

$$\begin{aligned}\frac{d\xi}{d\mu_4}&=r\left(\frac{P_1\mu_4^{2r}}{P}\right)^{\alpha}\frac{\nu_4}{\mu_4^{2r}},\\ \frac{d\nu_4}{d\mu_4}&=r\left(\frac{P}{P_1\mu_4^{2r}}\right)^{1-\alpha},\end{aligned}\tag{67}$$

and

$$\frac{P}{P_1\mu_4^{2r}}=\frac{1-\frac{\gamma-1}{2}\left(\frac{\nu_4}{\mu_4^{r}}\right)^2}{\xi},\tag{68}$$

while the transformation (65) becomes explicitly*

$$\begin{aligned}\mu_4&=\left(\frac{P_c}{P_1}\right)^{\frac{3-2\alpha}{2}}\mu^{2-\beta},\\ \nu_4&=\left(\frac{P_c}{P_1}\right)^{\frac{1}{2}}\nu.\end{aligned}\tag{69}$$

A solution of (67) is independent of the ballistic parameter P_c/P_1 and the pressure is simply proportional to P_1. Thus the parameter P_1 is fundamental to the limiting case $\vartheta=-1$ much as the Clemmow parameter P_3 is fundamental to the case $\vartheta=0$. The analogy goes even farther. For if we apply the similarity transformation (69) to the general case of equations (16), we obtain

$$\begin{aligned}\frac{d\xi}{d\mu_4}&=r\left(\frac{P_1}{P}\right)^{\alpha}\frac{\nu_4}{\mu_4^{1-r}},\\ \frac{d\nu_4}{d\mu_4}&=r\left(\frac{P}{P_1}\right)^{1-\alpha}\frac{1}{\mu_4^{1-r}},\end{aligned}\tag{70}$$

wherein

$$\frac{P}{P_1}=\frac{(1+\vartheta)\left(\frac{P_c}{P_1}\right)^{\frac{1}{2}}\mu_4^{r}-\vartheta\mu_4^{2r}-\frac{\gamma-1}{2}\nu_4^2}{\xi}.\tag{71}$$

Comparing (67), (68) and (70), (71) it is seen** that, except for notation, every solution with $P_c/P_1=0$ is the $\vartheta=-1$ solution. By (58), we note that $P_c/P_1=0$ corresponds to $P_3/P_c=\infty$ so that the $\vartheta=-1$ solution corresponds to $P_3/P_c=\infty$ while the $\vartheta=0$ solution corresponds to $P_3/P_c=0$. We have thus covered the complete spectrum of ballistic parameter values, $0\leqq\frac{P_3}{P_c}\leqq+\infty$, as well as the closed interval, $-1\leqq\vartheta\leqq1$, of geometric parameter values.

The general discussions in §§ 4 and 5 regarding the trajectory and pressure characteristics carry over to the $\vartheta=-1$ case where many of the results become sharper. [See for example (30).] For $\vartheta=-1$, the asymptotic (constant pressure) solution introduced in § 5 is obtained as an ordinary solution in the phase space! This may be seen as follows: From the general results in § 5, we infer that the pressure ratio P/P_1 as a function of ξ might appear as in Fig. 4. In particular, it rises to a maximum exceeding $(2/\gamma)^{\frac{1}{2-2\alpha}}$ and subsequently decays to this value as $\xi \to \infty$. However, as a function of $x = x_0\,\xi$, the pressure ratio P/P_1 curve will be compressed toward the initial point as $x_0 \to 0$, all other ballistic quantities, *i.e.*, P_1, being held fixed. The limiting form of P/P_1 as a function of x is, therefore, the constant function $P/P_1 \equiv (2/\gamma)^{\frac{1}{2-2\alpha}}$. It is easily verified that

---

* "Burnt" becomes $\mu_4=\left(\frac{P_c}{P_1}\right)^{\frac{3-2\alpha}{2}}$.

** A similarity transformation is required here.

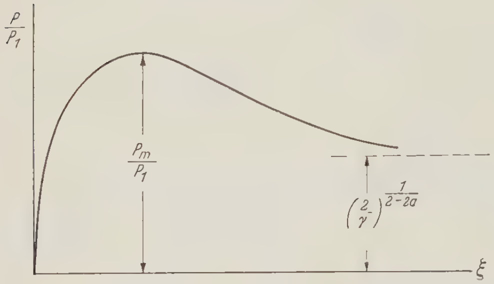

Fig. 4. Pressure curve for $\vartheta = -1$

$$x = \frac{1}{2}\,\frac{A}{M}\left(\frac{2}{\gamma}\right)^{\frac{1}{2-2\alpha}} P_1\,t^2,$$

$$v = \frac{A}{M}\left(\frac{2}{\gamma}\right)^{\frac{1}{2-2\alpha}} P_1\,t, \tag{72}$$

$$\sigma = \frac{B}{W}\left(\frac{2}{\gamma}\right)^{\frac{\alpha}{2-2\alpha}} P_1^{\alpha}\,t$$

is a constant pressure solution of equations (6) with $P/P_1 = (2/\gamma)^{\frac{1}{2-2\alpha}}$ and satisfies the initial condition, $x_0 = 0$. Since the piezometric efficiency for this solution is equal to one, we refer to $\vartheta = -1$ as the ideally progressive geometry. We note here that (72) is an isothermal solution with

$$\frac{T}{T_0} = \frac{1}{\gamma}.$$

It should be observed that any reasonable approximation to the $\vartheta = -1$ solution requires not only that ϑ be near -1, but according to equation (71), that the product $(1+\vartheta)\,(P_c/P_1)^{\frac{1}{2}}$ be small. This becomes important for any considerations of a double limit $\vartheta \to -1$, $x_0 \to 0$ in an attempt to approximate the constant pressure solution. For with other quantities fixed, $x_0 \to 0$ corresponds to $P_c \to \infty$!

10. The isothermal model and ballistic efficiencies

Because they are often simpler to treat, many ballistic theories are founded on isothermal expansions. Experience (mostly experimental and computational) has shown that the temperature drop of the gases during burning usually entails but a few per cent, so that the equation of state (3) very nearly describes an isothermal expansion. However, as the kinetic energy imparted to the shot (*i.e.*, the end product of the gun) is manifest in this very temperature drop, a strictly isothermal model would seem somewhat incongruous. In practice, the temperature drop is accounted for with an approximation to the temperature ratio T/T_0 by an empirical constant less than unity. This effectively reduces the force F of the propellant in proportion to the energy utilized. The fraction of potential energy of the propellant utilized by a gun is defined as the ballistic efficiency. Thus the validity of an isothermal solution is inextricably tied to consideration of the ballistic efficiency.

Clemmow [2] has shown that for a linear burning law, $\alpha = 1$, an isothermal expansion with a reduced force is justified up to maximum pressure. Here we extend this result to all cases and show that for propellants with progressive burning characteristics (*i.e.* $\vartheta < 0$) an isothermal expansion with reduced force is justified throughout burning.

From (21) we derive the inequality*

$$\xi \geqq \left(\frac{P_c}{P}\right)\frac{v^2}{2}, \tag{73}$$

which is valid up to maximum pressure**. Hence using (17), (73) implies that

$$\frac{P}{P_c} \leqq \frac{(1+\vartheta)\,\mu^{\beta}-\vartheta\,\mu^{2\beta}-\dfrac{\gamma-1}{2}\,v^2}{\dfrac{1}{2}\left(\dfrac{P_c}{P}\right)v^2}$$

or that

$$\frac{1}{2}v^2 \leqq \frac{(1+\vartheta)\,\mu^{\beta}-\vartheta\,\mu^{2\beta}}{\gamma} \tag{74}$$

holds up to maximum pressure. The ballistic efficiency is given by the ratio

$$\varepsilon_b = \frac{\frac{1}{2}(\gamma-1)\,v^2}{(1+\vartheta)\,\mu^{\beta}-\vartheta\,\mu^{2\beta}}, \tag{75}$$

so that (74) implies that ε_b does not exceed $\gamma-1/\gamma$. Because of (4), the latter merely expresses the fact that

$$\frac{T}{T_0} \geqq \frac{1}{\gamma}. \tag{76}$$

The value of γ is nearly constant for all propellants varying but a few per cent from the nominal value 1.25. Hence isothermal solutions utilizing, say, the constant value $\dfrac{1+\dfrac{1}{\gamma}}{2}$ as an approximation to $\dfrac{T}{T_0}$ will be universally accurate to about 10%, up to maximum pressure. Since the constant pressure solution obtained in § 9 is an isothermal solution with $T/T_0 = 1/\gamma$, (76) is the best possible result of universal character.

The bound (76) can be justified in general only up to maximum pressure. However from our results in § 5 (depicting the asymptotic pressure values for progressive burning propellants), we may obtain some useful extensions of (76). For example, the temperature ratio may be expressed as

$$\frac{T}{T_0} = \frac{\dfrac{P}{P_c}}{\dfrac{P}{P_c}+\dfrac{\gamma-1}{2}\,\dfrac{v^2}{\xi}}. \tag{77}$$

One extension is obtained by generalizing (73) to

$$\xi \geqq \left(\frac{P_c}{P_m}\right)\frac{v^2}{2}, \tag{78}$$

---

* Of course, the stronger result $\xi \geqq \left(\frac{P_c}{P}\right)\frac{v^2}{2}+1$ applies, but (73) is sufficient here.

** Here again we refer to the first local maximum of P.

which holds throughout. Then we have from (77)

$$\frac{T}{T_0} \geqq \frac{1}{1+(\gamma-1)\,\frac{P_m}{P}} \geqq \frac{P}{P_m}\,\frac{1}{\gamma} \tag{79}$$

throughout. Thus an isothermal solution is justified beyond maximum pressure in cases when the pressure P does not fall off too rapidly. Actually (21) indicates that for those cases when P approaches an asymptotic value as $\mu\to\infty$, *i.e.*, $\vartheta<0$, the ratio* $\frac{\frac{1}{2}\nu^2}{\xi}\to\frac{P}{P_c}$ so that (77) implies that $\frac{T}{T_0}\to\frac{1}{\gamma}$ as $\mu\to\infty$. In such cases we infer that (76) probably holds throughout!

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* Notice that both $\xi\to\infty$ and $\nu\to\infty$ as $\mu\to\infty$.

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The Equilibrium of Rods

A. E. Green

1. Introduction

In a recent paper J. L. Ericksen & C. Truesdell (1959) discuss the exact theory of stress and strain in rods and shells. They point out that stress resultants and couple resultants in shells may be regarded as derived quantities, defined in terms of the three-dimensional stress tensor, and that the equations of equilibrium for these resultants have been obtained from the three-dimensional equations of equilibrium for stresses by some writers. Ericksen & Truesdell also mention that an exact derivation of equations of equilibrium for rods based on the three-dimensional equations of equilibrium has not been worked out and the main purpose of this paper is to fill this gap. Using the explicit formulae for stress resultants and couples derived here, together with stress strain relations for large deformations, it may later be possible to obtain satisfactory expressions for these resultants in order to complete the theory of rods.

2. Notation and Formulae

We use the notation of Green & Zerna (1954) and summarize the relevant formulae here. Consider an elastic body* which changes from an undeformed to a deformed position and is then maintained in equilibrium by external forces. Points of the deformed body are defined by a general curvilinear system of coordinates ϑ^i where, for convenience, we use the notation

$$\vartheta^1 = \xi, \qquad \vartheta^2 = \eta, \qquad \vartheta^3 = \zeta. \tag{2.1}$$

The covariant and contravariant base vectors at each point of the deformed body are denoted respectively by $\boldsymbol{G}_i$, $\boldsymbol{G}^i$ with corresponding metric tensors G_{ij}, G^{ij}, latin indices having values 1, 2, 3. If a comma denotes partial differentiation then

$$\boldsymbol{G}_{i,j} = \Gamma^r_{ij}\,\boldsymbol{G}_r, \qquad \boldsymbol{G}^i_{,j} = -\,\Gamma^i_{jr}\,\boldsymbol{G}^r, \tag{2.2}$$

where Γ^r_{ij} are Christoffel symbols of the second kind and

$$\begin{aligned}
&\Gamma^r_{ij} = \Gamma^r_{ji} = G^{rs}\,\Gamma_{ijs},\\
&\Gamma_{ijs} = \Gamma_{jis} = \tfrac{1}{2}\left(G_{is,j} + G_{js,i} - G_{ij,s}\right),\\
&\Gamma_{irj} + \Gamma_{jri} = G_{ij,r},\\
&\Gamma^i_{ir} = \frac{1}{\sqrt{G}}\,\frac{\partial\sqrt{G}}{\partial\vartheta^r}, \qquad G = |G_{ij}|.
\end{aligned} \tag{2.3}$$

---

* The work presented here is, in fact, valid for a body which is not necessarily elastic.

The contravariant stress tensor is τ^{ij} and the stress vector $\boldsymbol{t}$ across a surface whose unit normal[*] is $\boldsymbol{u}$ is given by

$$\boldsymbol{t} = \frac{u_i \boldsymbol{T}_i}{\sqrt{G}}, \qquad \boldsymbol{T}_i = \sqrt{G}\,\tau^{ij}\,\boldsymbol{G}_j, \tag{2.4}$$

where

$$\boldsymbol{u} = u_i \boldsymbol{G}^i, \qquad G_{ij} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j. \tag{2.5}$$

If body forces are neglected[**] one form of the equations of equilibrium is

$$\boldsymbol{T}_{i,i} = 0, \qquad \boldsymbol{G}_i \times \boldsymbol{T}_i = 0, \tag{2.6}$$

where a comma denotes partial differentiation. From the second of equations (2.6) we get the symmetry property of the stress tensor.

3. Stress resultants and couples for a rod

We now suppose that the position vector $\boldsymbol{R}$ of any point of the deformed body is an analytic[***] function of the coordinates ξ, η in some range of values of ξ, η including $\xi=\eta=0$, so that

$$\boldsymbol{R} = \boldsymbol{R}^*(\zeta) + \xi \boldsymbol{A}_1(\zeta) + \eta \boldsymbol{A}_2(\zeta) + \cdots, \tag{3.1}$$

where $\boldsymbol{R}^*$, $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ are the values of $\boldsymbol{R}$, $\boldsymbol{G}_1$, $\boldsymbol{G}_2$ respectively when $\xi=\eta=0$. The equation $\xi=\eta=0$ defines a curve C with position vector $\boldsymbol{R}^*(\zeta)$ which we call the deformed rod. The boundary S of the deformed body is defined by an equation of the form

$$f(\xi, \eta) = 0 \tag{3.2}$$

such that $\zeta=$ constant are curved sections[****] of this body bounded by closed curves. From (3.1) and (2.2) we have

$$\boldsymbol{G}_i = \boldsymbol{A}_i + \xi \bar{\Gamma}^r_{i1} \boldsymbol{A}_r + \eta \bar{\Gamma}^r_{i2} \boldsymbol{A}_r + \cdots, \tag{3.3}$$

where $\bar{\Gamma}^r_{ij}$ denotes the value of Γ^r_{ij} when $\xi=\eta=0$, and $\boldsymbol{A}_3 = \boldsymbol{R}^*_{,3}$. Without loss of generality we may choose the coordinates ξ, η, ζ so that $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{A}_3$ are unit vectors. Hence

$$\boldsymbol{A}_1 \cdot \boldsymbol{A}_1 = \boldsymbol{A}_2 \cdot \boldsymbol{A}_2 = \boldsymbol{A}_3 \cdot \boldsymbol{A}_3 = 1. \tag{3.4}$$

If A_{ij}, A^{ij}, A denote the values of G_{ij}, G^{ij}, G respectively when $\xi=\eta=0$ it follows that

$$A_{ij} = \boldsymbol{A}_i \cdot \boldsymbol{A}_j, \qquad A_{11} = A_{22} = A_{33} = 1, \tag{3.5}$$

and recalling (2.3) we have

$$\bar{\Gamma}_{131} = \bar{\Gamma}_{232} = \bar{\Gamma}_{333} = 0, \tag{3.6}$$

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[*] The notation here for the unit normal vector differs from that in GREEN & ZERNA's book.

[**] Body forces may be included if required but for simplicity they are omitted here.

[***] We may assume a polynomial in ξ and η for $\boldsymbol{R}$ but this allows less freedom in the choice of the position vector in the undeformed body.

[****] If (3.1) is terminated after three terms then $\zeta=$ constant are plane sections of the deformed body.

where $\overline{\Gamma}_{ijr}$ denotes the value of Γ_{ijr} when $\xi = \eta = 0$. Also

$$\sqrt{G} = [\boldsymbol{G}_1 \boldsymbol{G}_2 \boldsymbol{G}_3], \qquad \sqrt{A} = [\boldsymbol{A}_1 \boldsymbol{A}_2 \boldsymbol{A}_3], \tag{3.7}$$

and we put

$$H = \sqrt{(G/A)} = 1 + \xi \overline{\Gamma}^i_{i1} + \eta \overline{\Gamma}^i_{i2} + \cdots. \tag{3.8}$$

The force across an element of area in a ζ-surface is $\boldsymbol{T}_3 \, d\vartheta^1 \, d\vartheta^2$ so that the stress resultant vector $\boldsymbol{n}$ across such a surface is

$$\boldsymbol{n} = \iint \boldsymbol{T}_3 \, d\xi \, d\eta \tag{3.9}$$

the integral being over a surface $\zeta =$ constant bounded by the surface (3.2). The couple resultant vector $\boldsymbol{m}$ across the surface $\zeta =$ constant, about the point $\xi = \eta = 0$ in the surface, is

$$\boldsymbol{m} = \iint (\boldsymbol{R} - \boldsymbol{R}^*) \times \boldsymbol{T}_3 \, d\xi \, d\eta. \tag{3.10}$$

If dS is an element of surface area in the surface (3.2) whose unit normal is $\boldsymbol{u}$ then

$$\boldsymbol{u} \, dS = u_i \boldsymbol{G}^i dS - (\boldsymbol{G}_1 \, d\vartheta^1 + \boldsymbol{G}_2 \, d\vartheta^2) \times \boldsymbol{G}_3 \, d\vartheta^3 = (\boldsymbol{G}^1 \, d\vartheta^2 - \boldsymbol{G}^2 \, d\vartheta^1) \sqrt{G} \, d\vartheta^3. \tag{3.11}$$

When surface forces are prescribed on (3.2) the force on dS is $\boldsymbol{t} \, dS$ and using (2.4) and (3.11) this becomes

$$\boldsymbol{t} \, dS = (\boldsymbol{T}_1 \, d\vartheta^2 - \boldsymbol{T}_2 \, d\vartheta^1) \, d\vartheta^3. \tag{3.12}$$

The surface forces over the surface (3.2) may be replaced by a force $\boldsymbol{p}$ and couple $\boldsymbol{q}$, measured per unit length of the rod C. An element of length of C is $ds = d\zeta$ so that, in view of (3.12),

$$\left.\begin{aligned} \boldsymbol{p} &= \oint (\boldsymbol{T}_1 \, d\eta - \boldsymbol{T}_2 \, d\xi), \\ \boldsymbol{q} &= \oint (\boldsymbol{R} - \boldsymbol{R}^*) \times (\boldsymbol{T}_1 \, d\eta - \boldsymbol{T}_2 \, d\xi), \end{aligned}\right\} \tag{3.13}$$

the integrals being taken around the curve

$$\zeta = \text{constant}, \quad f(\xi, \eta) = 0. \tag{3.14}$$

From (2.4), (3.3), (3.7) and (3.9) we obtain

$$\boldsymbol{n} = n^r \boldsymbol{A}_r \tag{3.15}$$

where

$$n^r = \sqrt{A} \iint H \tau^{i3} (\delta^r_i + \xi \overline{\Gamma}^r_{i1} + \eta \overline{\Gamma}^r_{i2} + \cdots) \, d\xi \, d\eta. \tag{3.16}$$

In (3.16) the double integrals are over the surface $\zeta =$ constant inside the surface (3.2). Also, from (2.4), (3.1), (3.7) and (3.10) we have

$$\boldsymbol{m} = \sqrt{A} \iint H \tau^{i3} \, (\delta^r_i + \xi \overline{\Gamma}^r_{i1} + \eta \overline{\Gamma}^r_{i2} + \cdots) \, (\xi \boldsymbol{A}_1 + \eta \boldsymbol{A}_2 + \cdots) \times \boldsymbol{A}_r \, d\xi \, d\eta. \tag{3.17}$$

But

$$\boldsymbol{A}_i \times \boldsymbol{A}_j = \varepsilon_{ijk} \boldsymbol{A}^k \tag{3.18}$$

where $\varepsilon_{ijk} = \pm \sqrt{A}$ according as i, j, k is an even or odd permutation of 1, 2, 3 and $\varepsilon_{ijk} = 0$ otherwise, so that (3.17) can be expressed in the form

$$\boldsymbol{m} = m_r \boldsymbol{A}^r \tag{3.19}$$

where

$$m_r = \sqrt{A} \iint H\, \tau^{i3} (\delta_i^s + \xi\, \bar{\Gamma}_{i1}^s + \eta\, \bar{\Gamma}_{i2}^s + \cdots)\, (\xi\, \varepsilon_{1sr} + \eta\, \varepsilon_{2sr} + \cdots)\, d\xi\, d\eta. \tag{3.20}$$

Finally, we express $\boldsymbol{p}$ and $\boldsymbol{q}$ in terms of vectors $\boldsymbol{A}_i$, $\boldsymbol{A}^i$ respectively in the forms

$$\boldsymbol{p} = p^i \boldsymbol{A}_i, \qquad \boldsymbol{q} = q_i \boldsymbol{A}^i. \tag{3.21}$$

4. Equations of equilibrium for rods

From (2.6) we have

$$\iint \boldsymbol{T}_{i,i}\, d\xi\, d\eta = 0, \qquad \iint (\boldsymbol{R} - \boldsymbol{R}^*) \times \boldsymbol{T}_{i,i}\, d\xi\, d\eta = 0, \tag{4.1}$$

the integrals being over a section $\zeta =$ constant of the deformed body bounded by the surface (3.2). The first of equation (4.1), together with (3.9) and (3.13), gives

$$\frac{\partial \boldsymbol{n}}{\partial s} + \boldsymbol{p} = 0. \tag{4.2}$$

Again, from (4.1), we see that

$$\begin{aligned} \frac{\partial}{\partial \zeta} \iint (\boldsymbol{R} - \boldsymbol{R}^*) \times \boldsymbol{T}_3\, d\xi\, d\eta - \iint (\boldsymbol{G}_3 - \boldsymbol{A}_3) \times \boldsymbol{T}_3\, d\xi\, d\eta\, + \\ + \iint [\{(\boldsymbol{R} - \boldsymbol{R}^*) \times \boldsymbol{T}_1\}_{,1} + \{(\boldsymbol{R} - \boldsymbol{R}^*) \times \boldsymbol{T}_2\}_{,2}]\, d\xi\, d\eta\, - \\ - \iint (\boldsymbol{G}_1 \times \boldsymbol{T}_1 + \boldsymbol{G}_2 \times \boldsymbol{T}_2)\, d\xi\, d\eta = 0 \end{aligned}$$

if we remember (3.1). Hence, using (3.9) and (3.10) this reduces to

$$\frac{\partial \boldsymbol{m}}{\partial \zeta} + \boldsymbol{A}_3 \times \boldsymbol{n} + \oint (\boldsymbol{R} - \boldsymbol{R}^*) \times (\boldsymbol{T}_1\, d\eta - \boldsymbol{T}_2\, d\xi) - \iint \boldsymbol{G}_i \times \boldsymbol{T}_i\, d\xi\, d\eta = 0$$

and in view of (3.13) and (2.6) we finally obtain

$$\frac{\partial \boldsymbol{m}}{\partial s} + \boldsymbol{A}_3 \times \boldsymbol{n} + \boldsymbol{q} = 0. \tag{4.3}$$

Equations (4.2) and (4.3) may be written in terms of components. From (3.15) we have

$$\frac{\partial \boldsymbol{n}}{\partial s} = \frac{\partial \boldsymbol{n}}{\partial \zeta} = \frac{\delta n^i}{\delta s} \boldsymbol{A}_i \tag{4.4}$$

where

$$\frac{\delta n^i}{\delta s} = \frac{\partial n^i}{\partial s} + \bar{\Gamma}_{j3}^i n^j, \qquad \boldsymbol{A}_{j,3} = \bar{\Gamma}_{j3}^i \boldsymbol{A}_i, \tag{4.5}$$

if we use (2.2) with $\xi = \eta = 0$. Similarly

$$\frac{\partial \boldsymbol{m}}{\partial s} = \frac{\delta m_i}{\delta s} \boldsymbol{A}^i \tag{4.6}$$

where

$$\frac{\delta m_i}{\delta s} = \frac{\partial m_i}{\partial s} - \bar{\Gamma}_{i3}^j m_j. \tag{4.7}$$

It follows from (3.21) and (4.4)—(4.7) that equations (4.2) and (4.3) have the alternative forms

$$\frac{\delta n^i}{\delta s} + p^i = 0, \qquad \frac{\delta m_i}{\delta s} + \varepsilon_{\alpha i} n^\alpha + q_i = 0 \tag{4.8}$$

where $\varepsilon_{12} = \sqrt{A}$, $\varepsilon_{21} = -\sqrt{A}$ and $\varepsilon_{\alpha i} = 0$ otherwise, α having the values 1, 2.

In the special case when the triad of unit vectors $\boldsymbol{A}_i$ is orthogonal

$$A_{12} = A_{23} = A_{31} = 0, \quad A = 1, \quad \boldsymbol{A}^i = \boldsymbol{A}_i, \quad A^{ij} = A_{ij}, \tag{4.9}$$

and using (2.3) and (3.6) we may write

$$\begin{aligned} \varkappa' &= \bar{\Gamma}^1_{33} = -\bar{\Gamma}^3_{13}, \\ \varkappa &= \bar{\Gamma}^3_{23} = -\bar{\Gamma}^2_{33}, \\ \tau &= \bar{\Gamma}^2_{13} = -\bar{\Gamma}^1_{23}, \\ \bar{\Gamma}^1_{13} &= \bar{\Gamma}^2_{23} = \bar{\Gamma}^3_{33} = 0. \end{aligned} \tag{4.10}$$

Equations (4.8) then reduce to a well-known classical form.

5. Remarks on further developments

In order to obtain explicit formulae for n^i and m_i from (3.16) and (3.20) we must use stress-strain relations. The position vector of any point of the undeformed body which corresponds to (3.1) is[*]

$$\boldsymbol{r} = \boldsymbol{r}^*(\zeta) + \xi\,\boldsymbol{a}_1(\zeta) + \eta\,\boldsymbol{a}_2(\zeta) + \cdots \tag{5.1}$$

if $\boldsymbol{r}$ is, for example, an analytic function of ξ, η in some range of values of these variables including $\xi = \eta = 0$. The vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ in (5.1) depend only on ζ and are not necessarily unit vectors. The vector

$$\boldsymbol{a}_3 = \frac{\partial \boldsymbol{r}}{\partial \zeta} \tag{5.2}$$

is not necessarily a unit vector but is tangent to the curve $\xi = \eta = 0$ which may be regarded as the undeformed rod c. From (5.1) we obtain expressions for base vectors $\boldsymbol{g}_i$, $\boldsymbol{g}^i$ in the undeformed body and hence metric tensors g_{ij}, g^{ij}. If the body is isotropic initially we then find the stress tensor τ^{ij} from formulae in GREEN & ZERNA. Difficulties remain in finding explicit values for n^i, m_i and this question is not considered further here. However, the analysis will be similar to that used by HAY (1942).

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[*] We may assume a polynomial form in ξ and η for $\boldsymbol{r}$ provided the expression (3.1) for $\boldsymbol{R}$ is, in general, an infinite series.

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Some Diffraction by Convex Bodies

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1. Introduction

The purpose of this article is to discuss a particular method of finding the solution of certain diffraction problems. Only scalar problems will be considered. The particular advantage of this method is that it leads quite readily to the exact solution as well as the asymptotic solution, with respect to large values of the wave number k, and that both the reflected and the diffracted wave are obtained simultaneously.

The particular configurations dealt with are the circular cylinder [*1*], the sphere [*2*], the parabolic cylinder [*3*], and as a special case of the latter the half plane [*4*], [*5*]. Each of these problems has been dealt with in the past, but it is hoped that this method will throw some new light on the nature of these solutions. The chief advantage is that all of these problems lend themselves readily to attack by this method. A complete bibliography is available in reference [*2*].

The problems involving convex surfaces have proved to be particularly obstinate and the method which was used to derive the asymptotic properties of the diffracted wave in all known cases was first developed by G. N. WATSON [*6*], [*7*]. This Watson transformation will prove to be fundamental in the present method.

Another method, which has been very successful in determining the leading terms in the asymptotic expansions, is due to J. B. KELLER [*8*], [*9*], [*10*]. It is an extension of the standard ray tracing technique, but diffracted rays are introduced. Unfortunately, no general proof of the validity of the method has as yet been furnished. The present treatment of the diffraction problems described above appears to have the following advantage: In the transition from wave optics to geometrical optics, the point (or points) on the diffracting body which are relevant in the theory of ray optics automatically appear in the asymptotic theory of the solutions of the rigorous diffraction problem.

It seems that the asymptotic formulas in the Appendices are not readily available in the literature.

2. Discussion of Method

The class of problems to be discussed can be formulated as follows: u must satisfy the conditions

1. $\Delta u + k^2 u = 0$,
2. $u + u_i = 0$ on surface $\mathfrak{S}$,
3. u satisfies the Sommerfeld radiation condition at ∞.

Here u_i is some incoming wave. In this article u_i will always be taken to be a plane wave, but as will be apparent the case of a spherical wave could be treated in an analogous manner. Similarly the boundary condition could be replaced by the condition

$$\frac{\partial}{\partial n}(u + u_i) = 0 \qquad \text{on surface } \mathfrak{S}.$$

The solution to this problem will now be assumed to be of the form

$$u = \int_{\mathfrak{S}} G f(s)\, ds.$$

Here G is the free space Green's function, *i.e.*, $H_0^{(1)}(kR)$ in two dimensional space and $\frac{e^{ikR}}{ikR}$ in three dimensional space, $\mathfrak{S}$ is the boundary surface, $f(s)$ a suitably chosen kernel and R is the distance from the point of observation to the surface. Evidently this integral satisfies the wave equation, as well as the radiation condition. $f(s)$ must now be so selected that the boundary condition is also satisfied. This leads to an integral equation, but fortunately, as will be seen subsequently, when suitable expansion theorems exist, the solution can be found quite readily.

$f(s)$ is then determined in the form of a line integral or sum, which can be rewritten as a line or contour integral. Thus u is ultimately determined in the form of a multiple integral. To this multiple integral a combination of the method of stationary phase and saddle point integration is then applied. It then develops that only two points in the space over which the integration is performed furnish sizable contributions. One of these corresponds to the reflected and the other to the diffracted wave. Furthermore, on the caustic, which in these problems corresponds to the boundary of the shadow region, the two points coalesce into a single point.

The chief difficulties arise in the evaluation of the multiple integrals. In order to find the points of stationary phase one must first represent the integrands by suitable asymptotic series. For the case of the circular cylinder and the sphere one has to use the Debye approximation to the Hankel functions [*11*]. For the parabolic cylinder their equivalents are derived in the appendices for the Weber or so-called parabolic cylinder functions. Once this has been done one can solve for the points of stationary phase. This will lead to several equations in as many unknowns, whose solution could be very cumbersome were it not for our knowledge of the basic laws of reflection. After this step the evaluation of the integrals is simple.

3. Diffraction by a Circular Cylinder

We shall assume that the incoming wave is a plane wave,

$$u_i = e^{ikr\cos\vartheta} = \sum_{-\infty}^{\infty} i^n J_n(kr)\, e^{in\vartheta},$$

and the scattered wave will be represented by

$$u_s = \int_{-\pi}^{\pi} H_0^{(1)}(kR)\, f(\vartheta)\, d\vartheta \tag{1}$$

where

$$R = \sqrt{r^2 + a^2 - 2ra\cos(\varphi - \vartheta)}\,.$$

Here (r, φ) denote the coordinates of the point of observation, a is the radius of the circular cylinder and ϑ describes the surface of the cylinder. (See Fig. 1.)

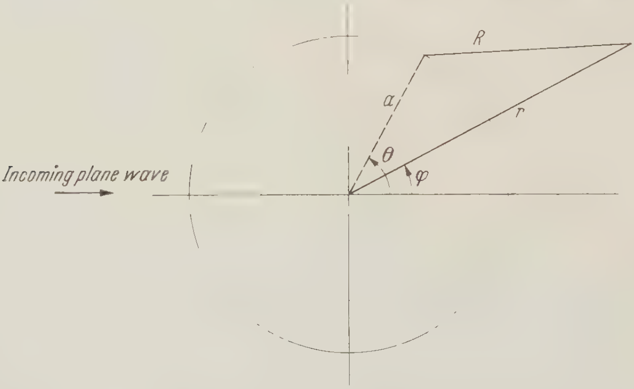

Fig. 1. Circular cylinder with incoming plane wave

Use of the addition theorem [*11*] for the Hankel functions allows one to show that

$$f(\vartheta) = -\frac{1}{2\pi} \sum_{-\infty}^{\infty} \frac{i^n e^{in\vartheta}}{H_n^{(1)}(ka)}\,.$$

The resultant u_s reduces to $-u_i$ for $r = a$.

One can now rewrite $f(\vartheta)$ as a contour integral:

$$f(\vartheta) = -\frac{1}{2\pi} \sum_{0}^{\infty} \varepsilon_n \frac{i^n \cos n\vartheta}{H_n^{(1)}(ka)} = -\frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\cos\nu\vartheta\, e^{-i\nu\pi/2}}{H_\nu^{(1)}(ka)\sin\nu\pi}\, d\nu\,, \tag{2}$$

$$\varepsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n \geq 1\,. \end{cases}$$

The contour $\mathfrak{C}$ is shown in Fig. 2.

Thus u_s is represented as a double integral, and one must use double stationary points to obtain an asymptotic evaluation of the integral. To do so one must express $H_\nu^{(1)}(ka)$ by means of the Debye approximation [*11*]

$$H_\nu^{(1)}(ka) \approx \sqrt{\frac{2}{\pi ka\cos\alpha}}\, e^{-i\pi/4}\, e^{ika[\cos\alpha + (\alpha - \pi/2)\sin\alpha]}$$

where $\sin\alpha = \nu / k a$. Similarly one must consider the asymptotic value of $H_0^{(1)}(kR)$, which is [*11*]

$$H_0^{(1)}(kR) \approx \sqrt{\frac{2}{\pi k R}}\, e^{i(kR-\pi/4)}.$$

Furthermore it will be assumed throughout this section that r is large, so that R can also be approximated by

$$R \approx r - a\cos(\varphi - \vartheta).$$

Under these assumptions one can now rewrite u_s in the form

$$u_s \approx \frac{1}{2\pi}\sqrt{\frac{a}{r}}\, e^{ikr}\int_{-\pi}^{\pi}\int_{\mathfrak{C}} e^{ig}(\cos\alpha)^{\frac{1}{2}}\, d\vartheta\, d\nu \tag{3}$$

where

$$g = -ka\left[\cos(\varphi - \vartheta) + \cos\alpha + \left(\alpha - \frac{\pi}{2}\right)\sin\alpha + \left(\frac{\pi}{2} + \vartheta\right)\sin\alpha - \pi\sin\alpha\right].$$

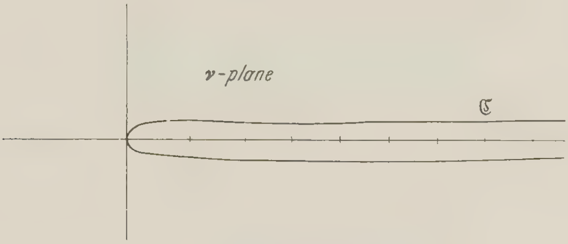

Fig. 2. Path of integration for integral (2)

In forming g, $\cos\nu\vartheta$ was replaced by $\frac{1}{2}e^{-i\nu\vartheta}$ and $\sin\nu\pi$ by $-\frac{1}{2i}e^{-i\nu\pi}$. These are the predominant terms, since as the subsequent analysis will show the region of interest in the ν-integral is the region $\frac{\operatorname{Re}\nu}{\operatorname{Im}\nu} > 0$. To find points of stationary phase we now set

$$\frac{\partial g}{\partial\vartheta} = -ka[\sin(\varphi - \vartheta) + \sin\alpha] = 0,$$

$$\frac{\partial g}{\partial\alpha} = -ka[\alpha + \vartheta - \pi]\cos\alpha = 0.$$

It is evident that the second of these two equations has two roots corresponding to the roots of the two factors. One now finds that there are two sets of roots

$$\alpha = \frac{\pi - \varphi}{2}, \qquad \vartheta = \frac{\pi + \varphi}{2},$$

$$\alpha = \frac{\pi}{2} + n\pi, \qquad \vartheta = \varphi \pm \frac{\pi}{2}.$$

The first of these corresponds to the reflected wave and the second to the diffracted wave. It can be observed that the former case yields the angle, which would be furnished by geometrical optics. In the latter case one finds that ϑ is independent of the ν-coordinate of the point of observation because ν was taken to be large. One can now express g in terms of Taylor series in the neighborhood of these points. Thus, to obtain the reflected wave one writes

$$g = 2ka\sin\frac{\varphi}{2} + \frac{ka}{2}\sin\frac{\varphi}{2}\left[\left(\vartheta - \frac{\pi+\varphi}{2}\right)^2 - \left(\alpha - \frac{\pi-\varphi}{2}\right)^2 - 2\left(\vartheta - \frac{\pi+\varphi}{2}\right)\left(\alpha - \frac{\pi-\varphi}{2}\right)\right].$$

Inserting this g in (3) and then evaluating the resultant Fresnel integrals asymptotically for large k leads to

$$u_{s,r} \approx -\sqrt{\frac{a\sin\frac{\varphi}{2}}{2r}}\, e^{ik\left(r - 2a\sin\frac{\varphi}{2}\right)}. \tag{4}$$

In order to obtain the diffracted wave one must proceed in a somewhat more careful manner. The point $\alpha = \pi/2$ is equivalent to letting $\nu = ka$, but the function

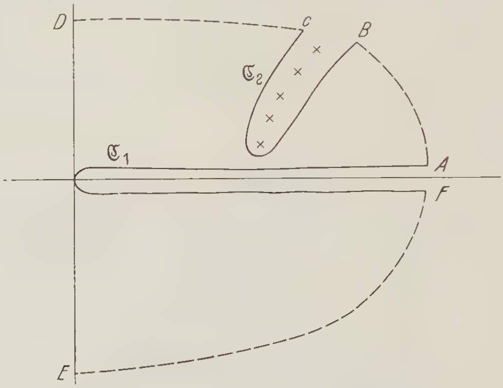

Fig. 3. Deformation of path of integration for integral (2)

$H_\nu^{(1)}(ka)$ which appears in the denominator of (2) has roots, as a function of ν near that point. It can be shown that these roots are given by the expression [5]

$$\nu_m = ka\left[1 + \frac{1}{2}\left(\frac{\frac{3\pi}{4}(4m+3)}{ka}\right)^{\frac{2}{3}} e^{i\pi/3}\right], \qquad m = 0, 1, 2, \ldots,$$

and that

$$\left[\frac{\partial H_\nu^{(1)}(ka)}{\partial \nu}\right]_{\nu=\nu_m} = (-)^m\, 2i\sqrt{\frac{2}{\pi}}\,\frac{1}{(ka)^{\frac{2}{3}}}\left[\frac{3\pi}{4}(4m+3)\right]^{\frac{1}{6}} e^{-i\pi/6}. \tag{5}$$

One can now deform the contour of the integral (2) from $\mathfrak{C}_1$ into $\mathfrak{C}_2$. (See Fig. 3.) To do so one must justify that the contributions along AB, CD, DE and EF vanish. That this is so along AB, CD and EF follows from an examination of the asymptotic properties of the integrand when ν is large. One can show that the contribution along DE vanishes because on that path the integrand is an odd function. One can now represent $f(\vartheta)$ as a residue summation over the roots of the Hankel functions:

$$f(\vartheta) = \frac{(ka)^{\frac{2}{3}} e^{i\pi/6}}{2i}\sqrt{\frac{\pi}{2}}\sum_{0}^{\infty}\frac{(-)^m \cos\nu_m\vartheta\, e^{-i\nu_m\pi/2}}{\left[\frac{3\pi}{4}(4m+3)\right]^{\frac{1}{6}}\sin\nu_m\pi}. \tag{6}$$

To obtain the diffracted wave we return to (1) and insert (6). We thus obtain, after replacing $\cos \nu_m \vartheta$ by its exponential equivalent, a sum of integrals of the form

$$\int e^{i[-k a \cos(\varphi-\vartheta) \pm \nu_m \vartheta]}\, d\vartheta .$$

These can be evaluated by stationary phase techniques, because the chief contribution will be due to the neighborhood of those points where

$$\frac{d}{d\vartheta}\left[-\cos(\varphi-\vartheta) \pm \vartheta\right] = 0 .$$

Furthermore, in consideration of the convergence of the series, one must take the range of integration from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ for the term with $+\vartheta$ and from $-\frac{3\pi}{2}$ to $\frac{\pi}{2}$ for the term with $-\vartheta$. The resultant integral can be expressed in terms of Airy functions. One can then write

$$\begin{aligned} u_{s,D} &= \frac{\pi (16 k a)^{\frac{1}{3}} e^{-5 i \pi/12} e^{i k r}}{\sqrt{k r}} \sum_0^{\infty} \frac{(-)^m \left[e^{i \nu_m \varphi} + e^{i \nu_m (2\pi - \varphi)}\right]}{\left[1 - e^{i 2 \pi \nu_m}\right] \left[\frac{3\pi}{4}(4m+3)\right]^{\frac{1}{6}}} \operatorname{Ai}\left\{-\left[\frac{3\pi}{8}(4m+3)\right]^{\frac{2}{3}}\right\} \\ &= \frac{-\pi (16 k a)^{\frac{1}{3}} e^{-5 i \pi/12} e^{i k r}}{i \sqrt{k r}} \sum_0^{\infty} \frac{(-)^m \cos \nu_m \varphi}{\sin \nu_m \pi \left[\frac{3\pi}{4}(4m+3)\right]^{\frac{1}{6}}} \operatorname{Ai}\left\{-\left[\frac{3\pi}{8}(4m+3)\right]^{\frac{2}{3}}\right\} \end{aligned} \tag{7}$$

where

$$\operatorname{Ai}(\alpha) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\alpha x + \frac{1}{3} x^3\right) dx .$$

4. Diffraction by a Sphere

We now turn to the problem of scattering by a sphere. The incoming wave will be expressed in spherical coordinates

$$u_i = e^{i k r \cos\vartheta} = \sum_0^{\infty} (2n+1)\, i^n \sqrt{\frac{\pi}{2 k r}}\, J_{n+\frac{1}{2}}(k r)\, P_n(\cos\vartheta) ,$$

and the scattered wave can be represented by

$$u_s = \int_0^{2\pi} \int_0^{\pi} \frac{e^{i k R}}{i k R} f(\vartheta, \varphi) \sin\vartheta \, d\vartheta \, d\varphi \tag{8}$$

where

$$R = \sqrt{r^2 + a^2 - 2 r a \cos\gamma} ,$$

$$\cos\gamma = \cos\vartheta \cos\vartheta_0 + \sin\vartheta \sin\vartheta_0 \cos(\varphi - \varphi_0) .$$

Here $(r, \vartheta_0, \varphi_0)$ represent the spherical coordinates of the point of observation and (a, ϑ, φ) the spherical coordinates of the scattering sphere.

Since the axis of symmetry of the plane wave was chosen to be the Z axis, one can expect that $f(\vartheta, \varphi)$ will be independent of φ. One can now expand $\frac{e^{i k R}}{i k R}$ into a series of characteristic functions of spherical waves [*11*]. Then, using the orthogonality of the spherical harmonics, one can show that

$$f(\vartheta) = -\frac{1}{2\pi}\sqrt{\frac{a\,k}{2\pi}}\sum_{0}^{\infty}\frac{(2n+1)\,i^n\,P_n(\cos\vartheta)}{H^{(1)}_{n+\frac{1}{2}}(k\,a)}$$

if

$$u_s = -u_i \quad \text{or} \quad r = a\,.$$

One can express $f(\vartheta)$ as a contour integral

$$f(\vartheta) = \sqrt{\frac{k\,a\,i}{2\pi}}\,\frac{1}{2\pi i}\int_{\mathfrak{C}}\frac{\nu\,P_{\nu-\frac{1}{2}}(\cos\vartheta)\,e^{-i\nu\pi/2}}{H^{(1)}_{\nu}(k\,a)\cos\nu\pi}\,d\nu \tag{9}$$

where $\mathfrak{C}$ encloses the roots of $\cos\nu\pi$ on the positive real axis. As with the circular cylinder we replace $H^{(1)}_{\nu}(ka)$ by the Debye approximation and the Legendre function by [*11*]

$$P_{\nu-\frac{1}{2}}(\cos\vartheta) \approx \sqrt{\frac{2}{\pi\nu\sin\vartheta}}\cos(\nu\vartheta-\pi/4) \qquad \vartheta \geqq \varepsilon > 0\,,$$

$$\approx J_0\left(\sqrt{\nu^2-\tfrac{1}{4}}\,\vartheta\right) \qquad \varepsilon > \vartheta \geqq 0\,.$$

Then (9) is put into (8), and the resultant triple integral can be evaluated by stationary phase techniques. The term R can be approximated by

$$R \approx r - a\cos\gamma = r - a(\cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos(\varphi-\varphi_0)\,,$$

and now the integrand contains the term e^{ig}, whose stationary points must be found. Then

$$g = -k\,a\left[\cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos(\varphi-\varphi_0) + \cos\alpha + \alpha\sin\alpha - (\pi-\vartheta)\sin\alpha\right],$$

and we have to solve the equations

$$\frac{\partial g}{\partial\vartheta} = 0\,, \qquad \frac{\partial g}{\partial\varphi} = 0\,, \qquad \frac{\partial g}{\partial\alpha} = 0\,.$$

One root will be near $\alpha = \pi/2$, where the integrand in (9) will have poles; this is the term corresponding to the diffracted wave. Corresponding to the reflected wave are found roots at

$$\varphi = \varphi_0\,, \qquad \vartheta = \frac{\pi+\vartheta_0}{2}\,, \qquad \alpha = \frac{\pi-\vartheta_0}{2}\,,$$

$$\varphi = \varphi_0 + \pi\,, \qquad \vartheta = \frac{\pi-\vartheta_0}{2}\,, \qquad \alpha = \frac{\pi+\vartheta_0}{2}\,.$$

The first of these points corresponds to the point of geometric reflection. The second one can be disregarded because $\alpha = \frac{\pi+\vartheta_0}{2}$ does not correspond to a point in the right half ν-plane. One can now expand g in a Taylor series about the first point, and one obtains

$$g = k\,a\Big[-2a\sin\frac{\vartheta_0}{2} + \frac{1}{2}\cos\frac{\vartheta_0}{2}\sin\vartheta_0(\varphi-\varphi_0)^2 +$$

$$+\frac{1}{2}\sin\frac{\vartheta_0}{2}\left(\vartheta-\frac{\pi}{2}-\frac{\vartheta_0}{2}\right)^2 - \frac{1}{2}\sin\frac{\vartheta_0}{2}\left(\alpha-\frac{\pi}{2}+\frac{\vartheta_0}{2}\right)^2 -$$

$$-\sin\frac{\vartheta_0}{2}\left(\vartheta-\frac{\pi}{2}-\frac{\vartheta_0}{2}\right)\left(\alpha-\frac{\pi}{2}+\frac{\vartheta_0}{2}\right)\Big].$$

The evaluation of (8) now leads to the expression for the asymptotic form of the reflected wave, namely

$$u_{s,R} \approx -\frac{a}{2R}\, e^{ik\left(r-2a\sin\frac{2}{\vartheta_0}\right)}. \tag{10}$$

To obtain the diffracted wave one can return to (9) and integrate over the residues due to the roots of the Hankel functions in the denominator. This is done in a manner completely analogous to the process carried out in the problem involving the circular cylinder. The result (9) can then be rewritten in the form

$$f(\vartheta) = \frac{-4(ka)^{\frac{7}{6}} e^{-i\pi/12}}{\sqrt{2\pi\sin\vartheta}} \sum_{0}^{\infty} \frac{(-)^m \nu_m^{-\frac{1}{2}} \cos(\nu_m - \pi/4)\, e^{-i\nu_m\pi/4}}{\left[\frac{3\pi}{4}(4m+3)\right]^{\frac{1}{6}} \cos\nu_m\pi}. \tag{11}$$

Here the ν_m denote the roots of the Hankel functions. (11) is then inserted in (8) and in order to evaluate the resultant integrals one can split the cosine term into its exponential components. Only the term $e^{-i\nu_m\vartheta+i\pi/4}$ contributes points of stationary phase, so that the other term may be neglected. In order to evaluate (11) one must evaluate integrals of the form

$$\int_0^{2\pi}\int_0^{\pi} e^{-i[ka\cos\gamma+\nu_m\vartheta]} \sqrt{\sin\vartheta}\, d\vartheta\, d\varphi. \tag{12}$$

The points of stationary phase are furnished by the roots of the equations

$$\frac{\partial}{\partial\varphi}[\cos\gamma+\vartheta] = 0$$

$$\frac{\partial}{\partial\vartheta}[\cos\gamma+\vartheta] = 0$$

which are $\begin{cases} \varphi = \varphi_0, & \vartheta = \vartheta_0 - \dfrac{3\pi}{2} \\ \varphi - \varphi_0 + \pi, & \vartheta = \vartheta_0 - \dfrac{\pi}{2} \end{cases}$. The phase must be chosen properly, to allow the series to converge. The φ-integral then becomes a Fresnel integral and the ϑ-integral an Airy integral. Actually the φ-integration can be carried out exactly in terms of Bessel functions, which are then approximated by their asymptotic series. It appears that little is gained in performing the exact integration. The above process is valid only if $0 < \varepsilon \leqq \vartheta_0 \leqq \pi - \varepsilon$. For $\vartheta_0 = 0$ or π, there is no point of stationary phase in φ. Upon performing the resultant integrations one obtains the diffracted wave.

$$\begin{aligned} u_{s,D} &= \frac{-\pi e^{ikr}(2ka)^{\frac{1}{3}} e^{-i5\pi/12}}{kr\sqrt{\sin\vartheta_0}} \sum_{0}^{\infty} \frac{(-)^m \nu_m^{-\frac{1}{2}}\left[e^{i\nu_m(\pi-\vartheta)} - i\, e^{i\nu_m(\pi+\vartheta_0)}\right]}{\cos\nu_m\pi\left[\frac{3\pi}{4}(4m+3)\right]^{\frac{1}{6}}} \times \\ &\qquad \times \operatorname{Ai}\left\{-\left[\frac{3\pi}{8}(4m+3)\right]^{\frac{2}{3}}\right\}, \\ &= \frac{-\pi e^{ikr}(16ka)^{\frac{1}{3}} e^{\quad 2\pi i.3}}{kr\sqrt{\sin\vartheta_0}} \sum_{0}^{\infty} \frac{(-)^m \nu_m^{\frac{1}{2}} e^{i\nu_m\pi} \cos(\nu_m\vartheta \quad \pi/4)}{\cos\nu_m\pi\left[\frac{3\pi}{4}(4m+3)\right]^{\frac{1}{6}}} \times \\ &\qquad \times \operatorname{Ai}\left\{-\left[\frac{3\pi}{8}(4m+3)\right]^{\frac{2}{3}}\right\}, \\ &\qquad 0 < \varepsilon \leqq \vartheta_0 < \pi - \varepsilon. \end{aligned} \tag{13}$$

To find the diffracted wave for $\vartheta_0=0$ we return to (12), where the φ integration is performed directly since now the integrand is independent of φ. The ϑ-integral is evaluated in the same manner as before, and we obtain

$$u_{s,D}=\frac{-\pi^{\frac{3}{2}}e^{ikr}(2ka)^{\frac{5}{6}}e^{-i5\pi/12}}{kr}\sum_{0}^{\infty}\frac{(-)^m\nu_m^{-\frac{1}{2}}e^{i\nu_m\pi}}{\cos\nu_m\pi\left[\frac{3\pi}{4}(4m+3)\right]^{\frac{1}{6}}}\,\mathrm{Ai}\left\{-\left[\frac{3\pi}{8}(4m+3)\right]^{\frac{2}{3}}\right\}. \tag{14}$$

The case $\vartheta_0=\pi$ yields a similar expression. To evaluate it one must return to (11), but the complex term $e^{i\nu_m\vartheta-\pi/4}$ must be used now, because the other term provides no point of stationary phase.

5. Diffraction by a Parabolic Cylinder

The technique applied in the earlier sections can be applied to this problem as well. But now one must operate with the appropriate functions and expansions in parabolic coordinates. It can be shown [*12*] that

$$\begin{aligned}H_0^{(1)}(kR)=\frac{\sqrt{2e^{-\pi i}}}{\pi}\frac{1}{2\pi i}\sum_{l=0}^{1}\int_{-\sigma-i\infty}^{-\sigma+i\infty}&\Gamma\left(-s+\frac{1}{4}+\frac{l}{2}\right)\Gamma\left(s+\frac{1}{4}+\frac{l}{2}\right)\times\\&\times E^{(l)}_{-2s-\frac{1}{2}}(2\sqrt{-ik\zeta_<})\,D_{-2s-\frac{1}{2}}(2\sqrt{-ik\zeta_>})\times\\&\times E^{(l)}_{2s-\frac{1}{2}}(2\sqrt{-ik\eta_<})\,D_{2s-\frac{1}{2}}(2\sqrt{-ik\eta_>})\,ds\end{aligned}\tag{15}$$

and

$$\begin{aligned}e^{ik[x\cos\alpha+y\sin\alpha]}=\frac{1}{\sqrt{2\pi\sin\alpha}}\frac{1}{2\pi i}\sum_{l=0}^{1}\int_{-\sigma-i\infty}^{-\sigma+i\infty}&\Gamma\left(-s+\frac{1}{4}+\frac{l}{2}\right)\Gamma\left(s+\frac{1}{4}+\frac{l}{2}\right)\times\\&\times(-)^l\left(\tan\frac{\alpha}{2}\right)^{-2s}E^{(l)}_{-2s-\frac{1}{2}}(2\sqrt{-ik\zeta})\,E^{(l)}_{2s-\frac{1}{2}}(2\sqrt{-ik\eta})\,ds\end{aligned}\tag{16}$$

where

$$0\leq\alpha\leq\pi,\qquad\sqrt{\zeta}\geqq0,\qquad0\leq|\sqrt{\eta}|<\infty,\qquad|\sigma|<\frac{1}{4}+\frac{l}{2}.$$

The parabolic coordinates are defined by

$$x=\zeta-\eta,\qquad y=\pm2\sqrt{\zeta\eta}$$

and in (15) the symbols $\zeta_<$, $\zeta_>$ are to be interpreted as the ζ coordinates of two points where $\zeta_<$ is the smaller and $\zeta_>$ the larger, and similarly for $\eta_<$, $\eta_>$. R is the distance between the two points. The function $D_\nu(Z)$ is a solution of the differential equation

$$y''+\left[\nu+\frac{1}{2}-\frac{Z^2}{4}\right]y=0$$

and can be defined in terms of the Whittaker function

$$D_\nu(Z)=2^{\nu/2}\left(\frac{Z^2}{2}\right)^{-\frac{1}{4}}W_{\nu/2+\frac{1}{4},\,\pm\frac{1}{4}}\left(\frac{Z^2}{2}\right).$$

The functions $E_\nu^{(l)}(Z)$ are solutions of the same equation and are defined by

$$E_\nu^{(l)}(Z)=\frac{(-)^l\,2^{-\frac{1+\nu}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{1-l-\nu}{2}\right)\{D_\nu(Z)+(-)^lD_\nu(-Z)\},\qquad l=0,1.$$

It can be seen from this definition that $E_\nu^{(0)}(Z)$ is an even function of Z, and $E_\nu^{(1)}(Z)$ is odd. The E and D functions play the same role in parabolic coordinates that the Bessel and Hankel functions play in polar coordinates, respectively. In other words, one associates standing waves with the E functions and cylindrical waves with the D functions. This may be observed in (15) and (16).

The scattering parabolic cylinder will be defined by $\zeta = \zeta_a$ so that the scattered wave will be representable as (see Fig. 4)

$$u_s = \int_{-\infty}^{\infty} H_0^{(1)}(kR)\, f(\eta)\, d\sqrt{-2ik\eta}. \quad (17)$$

Here R is the distance from the point of observation to the cylinder.

Again $f(\eta)$ must be so selected that on the surface u_s cancels out the incoming plane wave. This leads again to an integral equation for $f(\eta)$, which can be solved without too much difficulty by assuming a suitable form. Thus it appears reasonable to suppose that

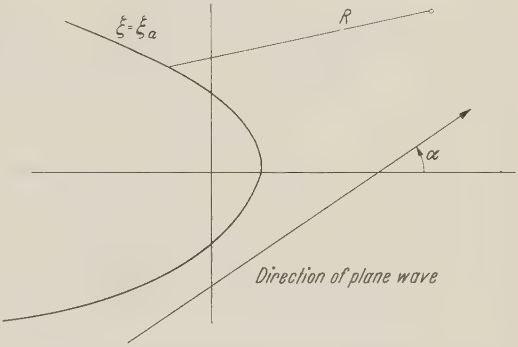

Fig. 4. Parabolic cylinder with incoming plane wave

$$f(\eta) = \frac{1}{2\pi i} \sum_{l=0}^{1} \int_{-\sigma-i\infty}^{-\sigma+i\infty} \frac{A_l(s)\left(\tan\frac{\alpha}{2}\right)^{-2s} E_{2s-\frac{1}{2}}^{(l)}(2\sqrt{-ik\eta})}{D_{-2s-\frac{1}{2}}(2\sqrt{-ik\zeta_a})}\, ds. \quad (18)$$

The E functions in the numerator are orthogonal, since one is even and the other odd, and in the denominator we require a D function, which will cancel the corresponding D function in the numerator of (15), so that when evaluating (17) for a point of observation on the surface the resultant expression will resemble the plane wave (16).

To determine the unknown terms $A_l(s)$ in (18) one inserts (18) and (15) in (17), and one has to use the integral formulas

$$\int_0^{\infty} E_{2s-\frac{1}{2}}^{(l)}(2\sqrt{-ik\eta_<})\, D_{2s-\frac{1}{2}}(2\sqrt{-ik\eta_>})\, E_{2\sigma-\frac{1}{2}}^{(l)}(2\sqrt{-ik\eta})\, d\sqrt{-2ik\eta}$$
$$= -\frac{\sqrt{2\pi}\, 2^{s+\frac{3}{4}} E_{2\sigma-\frac{1}{2}}^{(l)}(2\sqrt{-ik\eta_0})}{4(s-\sigma)\,\Gamma\left(-s+\frac{1}{4}+\frac{l}{2}\right)}, \qquad l = 0, 1.$$

Here η_0 is the η coordinate of the point of observation and

$$\left.\begin{matrix} \eta_< = \eta, & \eta_> = \eta_0 \\ \eta_< = \eta_0, & \eta_> = \eta \end{matrix}\right\} \text{ for the interval of integration } \left\{\begin{matrix} (0, \eta_0) \\ (\eta_0, \infty). \end{matrix}\right.$$

These formulas can easily be derived from general integration formulas given in the work by Buchholz [*12*].

(17) then reduces to a double line integral

$$u_s = \frac{\sqrt{2e^{-\pi i}}}{\pi} \frac{1}{(2\pi i)^2} \sum_{l=0}^{1} \int\limits_{-\sigma-i\infty}^{-\sigma+i\infty} \int\limits_{-\sigma-i\infty}^{-\sigma+i\infty} \Gamma\left(s+\frac{1}{4}+\frac{l}{2}\right) E^{(l)}_{-2s-\frac{1}{2}}\left(2\sqrt{-ik\zeta}\right) \times$$

$$\times \frac{D_{-2s-\frac{1}{2}}(2\sqrt{-ik\zeta_0})}{D_{-2\tau-\frac{1}{2}}(2\sqrt{-ik\zeta_a})} A_l(\tau)\left(\tan\frac{\alpha}{2}\right)^{-2\tau} \left(\frac{-\sqrt{2\pi}\,2^{\frac{3}{4}}}{2(s-\tau)}\right) E^{(l)}_{2\tau-\frac{1}{2}}\left(2\sqrt{-ik\eta_0}\right) ds\, d\tau\,.$$

One can now close the contour of integration for the variable s, using the appropriate asymptotic formulas [*12*], thereby evaluating the integral by means of its residue at the one singualr point $s=\tau$. One is then left, for $\zeta_0=\zeta_a$, with

$$u_s = \frac{\sqrt{2e^{-\pi i}}}{\pi} \sqrt{\frac{\pi}{2}} \frac{1}{2\pi i} \sum_{l=0}^{1} \int\limits_{-\sigma-i\infty}^{-\sigma+i\infty} \Gamma\left(s+\frac{1}{4}+\frac{l}{2}\right) E^{(l)}_{-2s-\frac{1}{2}}\left(2\sqrt{-ik\zeta}\right) \times$$

$$\times E^{(l)}_{+2s-\frac{1}{2}}\left(2\sqrt{-ik\eta_0}\right) A_l(s)\left(\tan\frac{\alpha}{2}\right)^{-2s} 2^{\frac{3}{4}+s}\, ds\,.$$

Since this is the value of the scattered wave on the surface, it must reduce to (16), and one obtains by comparison

$$A_l(s) = \frac{i(-)^l\,\Gamma\left(-s+\frac{1}{4}+\frac{l}{2}\right) 2^{-\frac{5}{4}}}{\sqrt{\sin\alpha}\; 2^s}$$

so that finally

$$f(\eta) = \frac{-i\,2^{-\frac{5}{4}}}{\sqrt{\sin\alpha}} \sum_{l=0}^{1} \frac{(-)^l}{2\pi i} \int\limits_{-\sigma-i\infty}^{-\sigma+i\infty} \frac{\Gamma\left(-s+\frac{1}{4}+\frac{l}{2}\right)\left(\tan\frac{\alpha}{2}\right)^{-2s} E^{(l)}_{+2s-\frac{1}{2}}\left(2\sqrt{ik\eta}\right)}{2^s\, D_{-2s-\frac{1}{2}}\left(2\sqrt{-ik\zeta_a}\right)}\, ds\,. \qquad (19)$$

Furthermore, one can show, using the linear dependence between the E and the D functions, that

$$f(\eta) = \frac{-1}{2\pi\sqrt{\sin\alpha}} \int\limits_{-\sigma-i\infty}^{-\sigma+i\infty} \Gamma\left(\frac{1}{2}-2s\right)\left(\tan\frac{\alpha}{2}\right)^{-2s} \frac{D_{2s-\frac{1}{2}}\left(2\sqrt{-ik\eta}\right)}{D_{-2s-\frac{1}{2}}\left(2\sqrt{-ik\zeta_a}\right)}\, ds\,. \qquad (20)$$

The next step is to combine (17) and (20) and to seek the stationary points in that double integral. To do so one must express the D functions in terms of appropriate asymptotic formulas, which are developed in the appendices. The details of this analysis are fundamentally the same as the corresponding steps in the analysis of the sphere and circular cylinder, but become much clumsier in the case of the reflected wave. Again two pairs of roots are found for the equations yielding the points of stationary phase. One pair corresponds to the reflected wave and the other to the diffracted wave. As was to be expected, the points yielding the diffracted wave are given by

$$\frac{\partial R}{\partial t} = -1, \qquad 2s = ik\zeta_a\,.$$

Here the parameter t represents the arclength of the parabola, and the important point is the point where the tangent ray from the point of observation to the parabola touches the surface. In the s integral we are again close to the residues of the integral. The reflected wave depends on the points where

$$\frac{\partial R}{\partial t} = -\cos\frac{\alpha-\vartheta}{2}, \qquad 2s = i\,k\,\zeta_a\left[1-\frac{\sin^2\frac{\alpha-\vartheta}{2}}{\sin^2\frac{\alpha+\vartheta}{2}}\right],$$

ϑ being the angular coordinate of the point of observation. The border of the shadow region and the illuminated region corresponds to $\alpha-\vartheta$, where the reflected and diffracted wave merge into one. By a direct but lengthy calculation one now finds that the reflected wave is given by

$$u_{s,R} = -2\sqrt{\frac{\varrho}{r}}\,\frac{\tan\frac{\alpha+\vartheta}{2}}{\sqrt{\sin\alpha}}\sqrt{\frac{\sin(\alpha-\vartheta)\sin\vartheta}{\csc\frac{\alpha+\vartheta}{2}-2\sec^2\frac{\alpha+\vartheta}{2}\sin\vartheta}}\times$$
$$\times\exp i\,k\left[r-\varrho\sin\frac{\alpha-\vartheta}{2}\left(\csc\frac{\vartheta+\alpha}{2}-\sin\frac{\alpha+\vartheta}{2}\right)+\varrho\sin^2\alpha\sin^2\vartheta\right] \qquad (21)$$

where ϱ is the distance of the point of reflection from the origin.

The diffracted wave can be obtained much more easily. To do so we return to (20) and find by a direct expansion over the residues of the D function in the denominator using the asymptotic formulas (A 8), (A 9), (A 13) that

$$f(\eta) = \frac{\sqrt{2\pi}\,e^{i\pi/6}(k\zeta_a)^{\frac{2}{3}}}{16\sqrt{\sin a}}\sum_0^\infty\frac{(-)^m\left(\tan\frac{\alpha}{2}\right)^{-2s_m}e^{2s_m}}{\left[\frac{3\pi}{8}(4m+3)\right]^{\frac{1}{6}}s_m^{\frac{1}{2}}\left(1+\frac{ik\eta}{2s_m}\right)^{\frac{1}{4}}}\times$$
$$\times\,\mathrm{e}^{ik\left[t+\delta_m\zeta_a\ln\left(\sqrt{\frac{\eta}{\zeta_a}}+\sqrt{1+\frac{\eta}{\zeta_a}}\right)\right]}. \qquad (22)$$

This expression is now inserted in (17), and the Hankel function is replaced by its asymptotic form. If that integral is evaluated by the method of stationary phase one finds that one is concerned with the point where the function

$$g = r + t$$

is stationary. It follows that

$$\frac{\partial g}{\partial\sqrt{\eta}} = \left(\frac{\partial r}{\partial t}+1\right)\frac{\partial t}{\partial\sqrt{\eta}}$$

vanishes at the point where $\frac{\partial r}{\partial t} = -1$, that is, the point of tangency. Furthermore

$$\frac{\partial^2 g}{\partial\sqrt{\eta}^2} = 0, \qquad \frac{\partial^3 g}{\partial\sqrt{\eta}^3} = \frac{2\zeta_a}{(\eta+\zeta_a)^{\frac{3}{2}}}$$

at that point. Again one is concerned with Airy integrals, and one obtains finally

$$u_{s,D} = \frac{-\pi e^{-i\pi\frac{11}{12}}}{256\sqrt{\sin a}\,(k\zeta_a)^{\frac{1}{6}}}\,e^{ikr}\sqrt{\frac{\varrho_0}{r}}\sum_0^\infty\frac{(-)^m\left(\tan\frac{\alpha}{2}\right)^{2s_m}e^{2s_m}}{\left[\frac{3\pi}{8}(4m+3)\right]^{\frac{1}{6}}(1+\delta_m)^{\frac{1}{2}}\left(1+\frac{ik\eta_0}{2s_m}\right)^{\frac{1}{4}}}\times$$
$$\times\,e^{ik\left[t_0+\delta_m\zeta_a\ln\left(\sqrt{\frac{\eta_0}{\zeta_a}}+\sqrt{1+\frac{\eta_0}{\zeta_a}}\right)\right]}\,\mathrm{Ai}\left\{-\left[\frac{3\pi}{8}(4m+3)\right]^{\frac{2}{3}}\right\}. \qquad (23)$$

The subscripts 0 refer to the point where the tangent ray from the point of observation to the parabola touches the surface, and ϱ_0 is the distance of that point from the origin.

6. Diffraction by a Half Plane

The solution to the problem of diffraction by a half plane can be obtained by returning to equations (17) and (20) and letting $\zeta_a \to 0$. Then, by use of

$$D_{-2s-\frac{1}{2}}(0) = \frac{\sqrt{\pi}\, 2^{-s-\frac{1}{4}}}{\Gamma(s+\frac{3}{4})},$$

(17) is transformed into

$$f(\eta) = \frac{-1}{2\pi\sqrt{\pi}\sin\alpha} \int\limits_{-\sigma-i\infty}^{-\sigma+i\infty} \Gamma\left(s+\frac{3}{4}\right) 2^{s+\frac{1}{4}} \left(\tan\frac{\alpha}{2}\right)^{-2s} \Gamma\left(\frac{1}{2}-2s\right) D_{2s-\frac{1}{2}}\left(2\sqrt{-ik\eta}\right) ds.$$

This integral can now be evaluated in terms of the residues of the integrand in the left half plane, so that

$$f(\eta) = \frac{-i}{\sqrt{\sin\alpha}} \sum_{0}^{\infty} \frac{(-)^N}{N!}\, 2^{-N-\frac{1}{2}} \left(\tan\frac{\alpha}{2}\right)^{N+\frac{3}{2}} \Gamma(2+2N)\, D_{-2N-2}\left(2\sqrt{-ik\eta}\right),$$

and, using the integral representation [*11*]

$$\Gamma(2+2N)\, D_{-2N-2}\left(2\sqrt{-ik\eta}\right) = e^{ik\eta} \int\limits_0^{\infty} e^{-2\sqrt{-ik\eta}\,t - \frac{t^2}{2}}\, t^{2N+1}\, dt,$$

one can sum the resultant exponential series to produce the closed form

$$f(\eta) = \frac{-i\left(\tan\frac{\alpha}{2}\right)^{\frac{3}{2}}}{\sqrt{2\pi\sin\alpha}}\, e^{ik\eta} \int\limits_0^{\infty} t\, e^{-2\sqrt{-ik\eta}\,t - \frac{t^2}{2}\sec^2\frac{\alpha}{2}}\, dt.$$

This integral can be evaluated directly, by means of the previous integral representation, so that, finally

$$f(\eta) = \frac{-i\sin\frac{\alpha}{2}\, e^{ik\eta\sin\frac{\alpha}{2}}}{2\sqrt{\pi}}\, D_{-2}\left(2\cos\frac{\alpha}{2}\sqrt{-ik\eta}\right). \tag{24}$$

Now one can rewrite (17) in the form

$$u_s = \frac{-i\sin\frac{\alpha}{2}}{2\sqrt{\pi}} \int\limits_{-\infty}^{\infty} H_0^{(1)}(kR)\, e^{ik\sin\frac{\alpha}{2}}\, D_{-2}\left(2\cos\frac{\alpha}{2}\sqrt{-ik\eta}\right) d\sqrt{-2ik\eta}. \tag{25}$$

This problem has been solved by similar techniques [*13*], [*14*], and in particular a very similar result was found in reference [*13*]. Well known asymptotic formulas can now be derived from (25). In particular the following representation [*11*] is helpful

$$D_{-2}(z) = e^{\frac{z^2}{4}} \left\{\sqrt{2}\, z \int\limits_{z/\sqrt{2}}^{\infty} e^{-t^2}\, dt - e^{-\frac{z^2}{2}}\right\}.$$

This can be represented in terms of the standard Fresnel functions.

Appendix I

The function $D_{-2s-\frac{1}{2}}(2\sqrt{-ik\zeta_a})$ has, as a function of s, roots in the left half plane. These can be located by finding a suitable asymptotic form for large k, which will be done by LANGER's method [*15*], [*16*]. The function $D_{-2s-\frac{1}{2}}(2\sqrt{-ik\zeta_a})$ satisfies the differential equation

$$\frac{d^2 y}{d z^2} - \left(2s + \frac{z^2}{4}\right) y = 0. \tag{A 1}$$

New dependent and independent variables are now introduced by the substitutions

$$Z = 2\sqrt{-2s}\,\varphi(\zeta), \quad \psi = \sqrt{\frac{d\varphi}{d\zeta}}, \quad y = \psi u,$$

leading to the equation

$$\frac{d^2 u}{d\zeta^2} + 16 s^2 (1 - \varphi_2)(\varphi')^2 u = 0. \tag{A 2}$$

φ, which as yet is undefined, is now determined by

$$\varphi' = \sqrt{\frac{-\zeta}{1-\varphi^2}}. \tag{A 3}$$

In this manner the equation is reduced to Airy's differential equation

$$\frac{d^2 u}{d\zeta^2} + 16 s^2 \zeta u = 0.$$

One now selects the solution that tends to 0 as $\zeta \to \infty$, namely

$$u = L\,\mathrm{Ai}[-\zeta(-16 s^2)^{\frac{1}{3}}] \tag{A 4}$$

where L is a suitable constant, independent of ζ.

$$\mathrm{Ai}(\alpha) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}x^3 + \alpha x\right) dx,$$

and for large negative arguments

$$\mathrm{Ai}(\alpha) \approx \frac{1}{\sqrt{\pi}\,\alpha^{\frac{1}{4}}} \sin\left\{\frac{2}{3}(-\alpha)^{\frac{3}{2}} + \frac{1}{4}\pi\right\}. \tag{A 5}$$

It is also known that for large z

$$D_{-2s-\frac{1}{2}}(z) \approx e^{-\frac{z^2}{4} - \left(2s + \frac{1}{2}\right)\ln z}. \tag{A 6}$$

From (A 3) one now finds that

$$\frac{2}{3}\zeta^{\frac{3}{2}} = \int_1^\varphi \sqrt{\varphi^2 - 1}\, d\varphi = \frac{\varphi\sqrt{\varphi^2-1} - \cosh^{-1}\varphi}{2},$$

so that for φ large

$$\frac{2}{3}\zeta^{\frac{3}{2}} \approx \frac{\varphi^2 - \ln 2\varphi}{2};$$

$$\text{for } \varphi \approx 1 \qquad \zeta \approx 2^{\frac{1}{3}}(\varphi - 1).$$

If one now assumes that φ is large, and combines (A 4), (A 5) and (A 6), one can evaluate L, and one finds that

$$L = 2^{\frac{3}{4}}\sqrt{\pi}\,e^{i\pi/4}\,e^{-(s-\frac{1}{4})\ln s - s\ln 2 + \pi i s};$$

also one finds, for $\varphi \approx 1$, that

$$D_{-2s-\frac{1}{2}}\left(2\sqrt{-ik\zeta_a}\right) \approx \frac{2^{\frac{1}{4}}\,e^{i\pi/4}\,e^{-(s-\frac{1}{4})\ln s - s\ln 2 + \pi i s}}{\left(s^2 - \frac{ik\zeta_a}{2}s\right)^{\frac{1}{4}}} \times$$
$$\times \sin\left\{-i\,\frac{8}{3}\,s\,2^{\frac{1}{2}}\left(\frac{ik\zeta_a}{2s} - 1\right)^{\frac{3}{2}} - \frac{\pi}{4}\right\}. \qquad \text{(A 7)}$$

One can now locate the roots of this function, and one finds

$$2s_m \approx ik\zeta_a\left\{1 + \left[\frac{\frac{3\pi}{8}(4m+3)}{k\zeta_a}\right]^{\frac{2}{3}} e^{\pi i/3}\right\}, \qquad m = 0, 1, 2, \ldots \qquad \text{(A 8)}$$

and

$$\frac{\partial}{\partial s} D_{-2s-\frac{1}{2}}\left(2\sqrt{-ik\zeta_a}\right)\Big]_{s=s_m} = \frac{-2^{\frac{9}{4}}\,e^{i\pi/12}\,e^{-(s_m-\frac{1}{4})\ln s_m - s_m\ln 2 + i\pi s}}{(k\zeta_a)^{\frac{2}{3}}} \times$$
$$\times (-)^m\left[\frac{3\pi}{8}(4m+3)\right]^{\frac{1}{8}}. \qquad \text{(A 9)}$$

Far away from the roots the approximation takes the form

$$D_{-2s-\frac{1}{2}}\left(2\sqrt{-ik\zeta_a}\right) \approx \frac{e^{i\pi/4}\,e^{-s\ln s + i\pi s}}{\sqrt{2}\,2^s\,(ik\zeta_a - 2s)^{\frac{1}{4}}} \times \qquad \text{(A 10)}$$
$$\exp\left\{\sqrt{ik\zeta_a(ik\zeta_a - 2s)} - 2s\ln\left(\sqrt{\frac{ik\zeta_a}{2s}} + \sqrt{\frac{ik\zeta_a}{2s} - 1}\right)\right\}.$$

Appendix II

One can analyze the function $D_{2s-\frac{1}{2}}(2\sqrt{-ik\eta})$ in a similar manner as was used in Appendix I. $D_{2s-\frac{1}{2}}(2\sqrt{-ik\eta})$ satisfies the differential equation

$$\frac{d^2y}{dz^2} + \left(2s - \frac{z^2}{4}\right)y = 0. \qquad \text{(A 11)}$$

New variables are now introduced by the substitutions

$$Z = 2\sqrt{-2s}\,\varphi(\zeta), \qquad \Psi = \sqrt{\frac{d\varphi}{d\tau}}, \qquad y = \Psi u,$$

and φ is defined by

$$\varphi' = \sqrt{\frac{\zeta}{1+\varphi^2}}$$

so that

$$\frac{2}{3}\zeta^{\frac{3}{2}} = \frac{\varphi\sqrt{1+\varphi^2} + \sin h^{-1}\varphi}{2}.$$

Thus (A 11) is reduced to the Airy differential equation

$$\frac{d^2u}{d\zeta^2} - 16s^2\zeta\,u = 0.$$

One can again combine (A 4), (A 5) and (A 6) to show that for large k

$$\begin{aligned}\Gamma\left(\frac{1}{2}-2s\right)D_{2s-\frac{1}{2}}\left(2\sqrt{-ik\eta}\right) &\approx \frac{\sqrt{\pi}\,e^{2s-s\ln s+\pi i s+\pi i/4}}{2^{s}(ik\eta+2s)^{\frac{1}{4}}}\times\\ &\times\exp\left\{\sqrt{ik\eta(ik\eta+2s)}+2s\ln\left(\sqrt{\frac{ik\eta}{2s}}+\sqrt{1+\frac{ik\eta}{2s}}\right)\right\}.\end{aligned} \qquad \text{(A 12)}$$

In particular one can now show the heavy dependence of (A 12) on the arclength t to the point (ζ_a, η) on the parabola from the origin, when S lies near the points (A 8). One can show that the arclength is given by

$$t=\int_0^{\eta}\sqrt{1+\frac{\zeta_a}{\eta}}\,d\eta=\sqrt{\eta(\eta+\zeta_a)}+\zeta_a\ln\left|\sqrt{\frac{\eta}{\zeta_a}}+\sqrt{1+\frac{\eta}{\zeta_a}}\right|,$$

so that (A 12) reduces to

$$\begin{aligned}\Gamma\left(\frac{1}{2}-2s_m\right)D_{2m-\frac{1}{2}}\left(2\sqrt{-ik\eta}\right) &\approx \frac{\sqrt{\pi}\,e^{2s_m-s_m\ln s_m+\pi i s_m+i\pi/4}}{2^{s_m}(ik\eta+2s_m)^{\frac{1}{4}}}\times\\ &\times\exp ik\left\{t+\delta_m\zeta_a\ln\left(\sqrt{\frac{\eta}{\zeta_a}}+\sqrt{1+\frac{\eta}{\zeta_a}}\right)\right\},\end{aligned} \qquad \text{(A 13)}$$

where

$$\delta_m=\left[\frac{\frac{3\pi}{8}(4m+3)}{k\,\zeta_a}\right]^{\frac{2}{3}}e^{i\pi/3}.$$

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Representation Theorems for Solutions of Linear Partial Differential Equations in Three Variables

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§ 1. Introduction

BERGMAN has discussed the properties of integral operators which generate solutions of linear partial differential equations with analytic coefficients in the case of 2 and 3 real variables [*1–6*]; see also KREYSZIG [*8, 9*]. In the case of 3 variables they consider the equation

$$\Delta\psi + F(r^2)\,\psi = 0, \tag{1}$$

where F is an entire function of $r^2 = x^2 + y^2 + z^2$, $X = (x, y, z)$ a point in three dimensional x, y, z-space $E_3 - x, y, z$ real[1]. The methods and results obtained in the theory of harmonic functions of 3 variables can be utilized in the study of solutions of (1), but first the theory of harmonic functions of three variables had to be developed as a basis for the theory of functions satisfying (1). See [*1, 2, 3*]. These results on harmonic functions are of interest in themselves.

The Whittaker-Bergman operator ([*1*], [*5*] p. 467):

$$H(X) = B(f, \mathfrak{L}, X_0) = (1/2\pi i) \int_{\mathfrak{L}} f(u, \zeta)\, d\zeta/\zeta, \tag{2}$$

transforms an analytic function f (called the B_3-associate[2] of H) of two complex variables u and ζ into a harmonic function of three variables for X in the neighborhood of a point X_0. Here

$$\begin{aligned} u &= u(X, \zeta) \equiv x + Z\zeta + Z^*\zeta^{-1}, \\ Z &= \tfrac{1}{2}(iy + z), \qquad Z^* = \tfrac{1}{2}(iy - z), \end{aligned} \tag{3}$$

and $\mathfrak{L}$ is a closed rectifiable curve in the ζ-plane. The representation (2) at a point X_0 where f is regular is valid at first in the small (that is, in a sufficiently small neighborhood of the point X_0). BERGMAN has shown how we can use the correspondence $f(u, \zeta) \to H(X)$ for the study of harmonic functions in the large, in the case where $f(u, \zeta)$ is a rational or algebraic function or the integral of an algebraic function. The harmonic function defined by (2) is not regular throughout E_3. For example, if f is a rational function, the space E_3 is subdivided into a finite number of regions (called domains of associations) which depend upon f and the curve $\mathfrak{L}$ in the ζ-plane, such that within each region (2) defines a regular harmonic function. As the point X moves from one region to another, singular points of f enter or leave the interior of $\mathfrak{L}$, and a new harmonic function is defined.

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[1] Recently BERGMAN has investigated also equations $\Delta\psi + F(y, z)\,\psi = 0$, where F is an entire function of y, z.

[2] We note that the B_3-associate of a given harmonic function depends upon the choice of the coordinate system. See [*5*], p. 467.

One problem in the study of integral operators is to obtain integral relations which generalize the residue and Abel theorems in the theory of analytic functions of one complex variable [5]. In this case we consider harmonic vectors $\vec{H} = (H_1, H_2, H_3)$, that is, vector fields in three-dimensional space which satisfy the system of equations[3]

$$\operatorname{div} \vec{H} = 0, \qquad \operatorname{curl} \vec{H} = 0 \tag{4}$$

(see [5], p. 488). In particular we consider harmonic vectors,

$$\vec{H}(X) = \vec{B}(f, \mathfrak{L}, X_0), \tag{5}$$

whose first component H_1 is given by (2) and[4]

$$\begin{aligned} H_2(X) &= \frac{1}{2\pi i} \int_{\mathfrak{L}} f(u,\zeta)\, \frac{1}{2}\, i(\zeta + \zeta^{-1})\, \frac{d\zeta}{\zeta}, \\ H_3(X) &= \frac{1}{2\pi i} \int_{\mathfrak{L}} f(u,\zeta)\, \frac{1}{2}\, (\zeta - \zeta^{-1})\, \frac{d\zeta}{\zeta}. \end{aligned} \tag{6}$$

Let

$$\Lambda_0(X, X_0) = \int_{\mathfrak{I}_0^1} \vec{H}(X) \cdot d\vec{X}, \tag{7}$$

where $\mathfrak{I}_0^1$ is a rectifiable curve lying in the regularity domain of H with initial point X_0 and end-point X. In case f is a rational B_3-associate:

$$f(u,\zeta) = \frac{P(u,\zeta)}{Q(u,\zeta)},$$

P and Q being polynomials in u and ζ, BERGMAN [5] has shown that there exist simple relations between a linear combination of the transcendental functions $\Lambda_0(X, X_0)$ and a linear combination of certain integrals of algebraic functions of one complex variable,

$$\int_{\mathfrak{L}} \left(\frac{P(u,\zeta)}{\zeta\, \partial Q(u,\zeta)/\partial\zeta} \right)_{u = u_\nu(\zeta)} d\zeta,$$

representing a different class of transcendental functions. In these relations the arguments x, y, z of $\vec{H}$ and argument of the function of one complex variable are connected by the algebraic equation

$$Q[u(X,\zeta), \zeta] = 0. \tag{8}$$

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[3] It should be noted that these vectors represent only a part of the class of vectors whose three components are harmonic.

[4] We note that the components H_2 and H_3 to a given H_1 are determined only within $\operatorname{Re}[g(y+iz)]$ and $-\operatorname{Im}[g(y+iz)]$, respectively. Here g is an analytic function of the complex variable $y+iz$, regular at the origin.

In this paper these relations between integrals are generalized to the case where f is an algebraic function. In § 2 a relation is obtained between a linear combination of certain transcendental functions increased by period functions [*cf.* (2.21)][5] and integrals of certain algebraic functions of one complex variable, also, increased by period functions (2.27). In § 3 examples are given to illustrate the general theory. In § 4 an analogous result is obtained in the case of rational B_3-associates for solutions of equation (1).

Notation:

In the following, bold-faced Italic characters denote vectors, while German special characters denote manifolds. Superscripts on manifolds lying in E_3 denote their real dimensions. The notation $E[\ldots]$ means the set of points satisfying the condition in the brackets.

§ 2. Integral theorem for harmonic vectors with algebraic associates

1. The domain of regularity of a function with an algebraic B_3-associate. The first step in our investigations consists in determining the domain where certain harmonic functions are defined by (1.2), in the case of an algebraic B_3-associate f (see Theorem 2.1). Let f be an algebraic function

$$f(u, \zeta, S) = \frac{p(u, \zeta, S)}{q(u, \zeta)}, \tag{1}$$

where p is a polynomial in u, ζ and S and q in u and ζ; u, ζ and S are connected by an irreducible algebraic equation

$$v(u, \zeta, S) = \sum_{\nu=0}^{n} p_\nu(u, \zeta)\, S^\nu = 0; \tag{2}$$

p_ν are polynomials in u and ζ. Under the transformation (1.3), v goes into

$$v(u, \zeta, S) = \zeta^{-p_0} V(\zeta, X, S) = \zeta^{-p_0} \sum_{\nu=0}^{n} q_\nu(\zeta, X)\, S^\nu \tag{3}$$

(here p_0 is a positive integer and q_ν are polynomials in ζ and X). For fixed X, $V(\zeta, X, S) = 0$, considered as an equation in S defines for $q_n \neq 0$ a Riemann surface $\mathfrak{R}(X)$ of n sheets and of genus $\varrho(X)$, lying over the ζ-plane. The genus $\varrho(X) = \varrho$ is in general (that is, except for some values of X noted below) independent of X. The branch points $e_j(X)$ of $\mathfrak{R}(X)$ which lie in the finite ζ-plane are included among the roots of $q_n(\zeta, X) = 0$ and roots of the discriminant $d(\zeta, X) = 0$ of (3). We assume $d(\zeta, X) \not\equiv 0$. The topological structure of $\mathfrak{R}(X)$ is different from that described above in the following cases:

i. If for any X, $q_n(\zeta, X)$ is zero for all ζ, then $\mathfrak{R}(X)$ has less than n sheets. This situation occurs if q_n has a factor, say $b(X)$, independent of ζ. Since $b(X)$ is a complex function, $b(X) = 0$ represents in general two equations $\operatorname{Re} b(X) = 0$, $\operatorname{Im} b(X) = 0$ and hence defines a curve in E_3. Since $q_n(\zeta, X)$ is a polynomial in ζ and X, it can have only a finite number of factors independent of ζ, such as $b(X)$. Hence there are at most a finite number of curves in E_3 such that for X on these curves, the surface $\mathfrak{R}(X)$ has less than n sheets. Let $\mathfrak{S}_1^1$ be the set of points in E_3 lying in these curves.

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[5] (2.21) means Formula (21) of § 2.

ii. The discriminant $d(\zeta, X)$ is obtained by eliminating S from $V(\zeta, S, X)=0$ and $V_S(\zeta, S, X)=0$ and hence is an entire rational function in ζ and X. Thus there are at most a finite number of curves in E_3 along which $d(\zeta, X)$ is zero for all ζ. Along such curves $V(\zeta, S, X)$ has a repeated irreducible factor involving S, *cf.* [7], and the structure of $\mathfrak{R}(X)$ changes. Call $\mathfrak{S}_2^1$ this set of points.

iii. Two branch points coincide if $q_n(\zeta, X)=0$ or $d(\zeta, X)=0$ have multiple roots, that is, two branch points coincide along the set of points X for which $q_n(\zeta, X)=0$ and $\partial q_n(\zeta, X)/\partial\zeta=0$ or $d(\zeta, X)=0$ and $\partial d(\zeta, X)/\partial\zeta=0$. Such a set of points ($\mathfrak{S}_3^1$) usually lies on curves in E_3 for if ζ is eliminated from, say, $q_n(\zeta, X)=0$ and $\partial q_n(\zeta, X)/\partial\zeta=0$, the resulting function is in general a complex function of X and hence defines a curve. Let

$$\mathfrak{S}^1 = E\left[X \middle| X\in \mathfrak{S}_1^1 \cup \mathfrak{S}_2^1 \cup \mathfrak{S}_3^1\right]. \tag{4}$$

Under transformation (1.3) suppose that $f(u, \zeta, S)$ goes into

$$F(\zeta, S, X) = \frac{P(\zeta, S, X)}{Q(\zeta, X)}, \tag{5}$$

where

$$Q(\zeta, X) = \zeta^{p_1} q(u, \zeta), \tag{6}$$

p_1 an integer. The function $F(\zeta, S, X)$ is determined in a Riemann domain $\mathfrak{R}^3$, that is, in a domain consisting of n space sheets. If we denote by $\mathfrak{R}_1^3(\zeta)$ the Riemann domain of the function $F(\zeta, S, X)$, ζ fixed, then for every X except for X belonging to $\mathfrak{S}^1=\mathfrak{S}_1^1\cup\mathfrak{S}_2^1\cup\mathfrak{S}_3^1$, the domain has the same topological structure and therefore for $X\in\mathfrak{R}^3-\mathfrak{S}^1$ *the genus* $\varrho_1(\zeta)=\varrho$ *of* $\mathfrak{R}_1^3(\zeta)$ *is constant.*

Let $\mathfrak{L}_0$ be a rectifiable curve in the ζ-plane, not going through the origin, with initial and end-points ζ_1 and ζ_2 respectively and consider[6]

$$\int\limits_{\mathfrak{L}_0} F(\zeta, S, X)\frac{d\zeta}{\zeta}. \tag{7}$$

We determine the domain of regularity of (7). For fixed $\zeta\neq 0$ the equation $Q(\zeta, X)=0$ or $q(u, \zeta)=0$, which may be written in the form

$$a_0(\zeta)\prod_{\nu=1}^{s}\left[u-u_\nu(\zeta)\right]=0, \qquad u=u(\zeta, X)$$

[*cf.* (1.3)], or

$$u=u_\nu(\zeta) \qquad (\nu=1, \ldots, s),$$

that is

$$u(\zeta, X)=u_\nu(\zeta) \qquad (\nu=1, \ldots, s)$$

defines a finite set of straight lines $\mathfrak{N}^1(\zeta)$ in E_3. For $\zeta\in\overline{\mathfrak{L}}_0$, $\mathfrak{N}^1(\zeta)$ trace a finite set of surfaces

$$\mathfrak{N}^2 = \bigcup_{\zeta\in\overline{\mathfrak{L}}_0}\mathfrak{N}^1(\zeta) = E\left[X \middle| Q(\zeta, X)=0,\ \zeta\in\overline{\mathfrak{L}}_0\right]. \tag{8}$$

For $X\in\mathfrak{N}^2$ the integrand of (7) is undefined.

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[6] We note that by changing $\mathfrak{L}_0$ the function given by (7) can be analytically continued. One of the problems in our approach is to study the properties of the function obtained by analytic continuation.

Similarly for fixed ζ the equations $q_n(\zeta, X)=0$ and $d(\zeta, X)=0$ define in general curves $\mathfrak{B}_1^1(\zeta)$ and $\mathfrak{B}_2^1(\zeta)$ respectively and as ζ traces $\bar{\mathfrak{L}}_0$ they trace surfaces in E_3:

$$\mathfrak{B}^2 = E[X \,|\, q_n(\zeta, X) = 0 \quad \text{or} \quad d(\zeta, X) = 0,\ \zeta \in \bar{\mathfrak{L}}_0]. \tag{9}$$

For $\zeta \in \bar{\mathfrak{L}}_0$, $Q(\zeta, X)=0$ only at those points X which by (8) belong to $\mathfrak{N}^2$. For other values of $X \notin \mathfrak{N}^2 \cup \mathfrak{S}^1$, $F(\zeta, S, X)$ is a regular function of ζ and hence (7) exists at such points.

Hence follows

Theorem 2.1. *Let $F(\zeta, S, X)$ be the rational function defined by* (5); *ζ, S and X being connected by $V(\zeta, S, X)=0$. Then the integral* (7) *represents a function which is defined on the Riemann domain $\mathfrak{R}^3$ at all points except when $X \in \mathfrak{N}^2 \cup \mathfrak{S}^1$.*

2. Behavior in the large of functions $H_1(X)$ with algebraic associates. According to a result due to Weierstrass [*10*], $F_0 = F/\zeta$ can be expressed for all $X \in \mathfrak{R}^3 - \mathfrak{S}^1$ in the form

$$\begin{aligned} F_0(\zeta, S, X) = \sum_{\nu=1}^{r(X)} c_\nu(X)\, G(\zeta_\nu, S_\nu; \zeta, S; X) + \frac{d}{d\zeta} \sum_{\nu=1}^{r(X)} R_\nu(\zeta, S, X) - \\ - \sum_{\alpha=1}^{\varrho} [f'_\alpha(X)\, G_\alpha(\zeta, S, X) - f_\alpha(X)\, G'_\alpha(\zeta, S, X)], \end{aligned} \tag{10}$$

where (ζ_ν, S_ν) are the positions of the poles of $F_0(\zeta_t, S_t, X)\, \dfrac{d\zeta}{dt}\, t$ on the Riemann surface $\mathfrak{R}(X)$ (t is the local uniformizing variable on $\mathfrak{R}(X)$), $c_\nu(X)$ is the residue of F_0 at (ζ_ν, S_ν) and G_α, G'_α, G are Weierstrass integrands of first, second and third kind respectively. The functions $c_\nu(X)$, $f_\alpha(X)$ and $f'_\alpha(X)$ are algebraic since they are obtained by algebraic operations from algebraic functions; $R_\nu(\zeta, S, X)$ is a rational function of ζ, S and X, *i.e.*, it is an algebraic function of ζ and X.

Let

$$\begin{array}{ll} 2\omega_{\alpha\beta}(X), & 2\omega'_{\alpha\beta}(X) \\ 2\eta_{\alpha\beta}(X), & 2\eta'_{\alpha\beta}(X) \\ \Omega_\beta(\zeta_\nu, S_\nu, X), & \Omega'_\beta(\zeta_\nu, S_\nu, X) \end{array} \tag{11}$$

$(\alpha, \beta = 1, \ldots, \varrho;\ \nu = 1, \ldots, r/(X))$ represent systems of primitive periods of Abelian integrals of first, second and third kind respectively. If $c_\beta(X)$ and $c'_\beta(X)$ are the cuts and conjugate cuts introduced on $\mathfrak{R}(X)$ to transform it into a simply connected surface $\mathfrak{R}'(X)$, then

$$\begin{array}{ll} 2\omega_{\alpha\beta}(X) = I_\alpha(c_\beta(X), X), & 2\omega'_{\alpha\beta}(X) = I_\alpha(c'_\beta(X), X) \\ 2\eta_{\alpha\beta}(X) = I'_\alpha(c_\beta(X), X), & 2\eta'_{\alpha\beta}(X) = I'_\alpha(c'_\beta(X), X) \\ \Omega_\beta(\zeta_\nu, S_\nu, X) = I(c_\beta(X); \zeta_\nu, S_\nu; X), & \Omega'_\beta(\zeta_\nu, S_\nu, X) = I(c'_\beta(X), \zeta_\nu, S_\nu; X), \end{array} \tag{12}$$

where

$$\begin{aligned} I_\alpha(C, X) &= \int_C G_\alpha(\zeta, S, X)\, d\zeta \\ I'_\alpha(C, X) &= \int_C G'_\alpha(\zeta, S, X)\, d\zeta \\ I(C; \zeta_\nu, S_\nu; X) &= \int_C G(\zeta_\nu, S_\nu; \zeta, S; X)\, d\zeta \end{aligned} \tag{13}$$

for C an open curve joining two points of $\mathfrak{R}(X)$ and not going through a branch point.

Now let $\mathfrak{L}$ be a closed regular curve on $\mathfrak{R}(X)$ which does not meet the origin, the branch points of $\mathfrak{R}(X)$ or the poles of $F(\zeta, S, X)$. Assuming that $\mathfrak{L}$ intersects the cuts $c_\beta(X)$, $c'_\beta(X)$ only a finite number of times and at each intersection crosses the cut (*cf.* [7]), we define $n_\beta(X)$ as the difference between the number of times $\mathfrak{L}$ crosses $c_\beta(X)$ from the negative to the positive side and the number of times it crosses in the opposite direction; similarly define an integer $n'_\beta(X)$ with respect to $c'_\beta(X)$. Then the Abelian integrals of the first and second kind over $\mathfrak{L}$ give

$$\begin{aligned} I_\alpha(\mathfrak{L}, X) &= \sum_{\beta=1}^{\varrho} [2n_\beta(X)\,\omega_{\alpha\beta}(X) + 2n'_\beta(X)\,\omega'_{\alpha\beta}(X)] \equiv S_{\alpha\varrho}(X) \\ I'_\alpha(\mathfrak{L}, X) &= \sum_{\beta=1}^{\varrho} [2n_\beta(X)\,\eta_{\alpha\beta}(X) + 2n'_\beta(X)\,\eta'_{\alpha\beta}(X)] \equiv S'_{\alpha\varrho}(X). \end{aligned} \tag{14}$$

If loops $l_\nu(X)$ are introduced around the points (ζ_ν, S_ν) on the surface $\mathfrak{R}'(X)$ to remove the points at which the integrand $F_0(\zeta, S, X)$ has poles, the resulting surface $\mathfrak{R}''(X)$ is also simply-connected (*cf.* [7]). If $n''_\nu(X)$ is the difference in the number of times $\mathfrak{L}$ intersects $l_\nu(X)$ from the negative to the positive side and vice versa, then the integral of the third kind gives

$$\begin{aligned} I(\mathfrak{L}; \zeta_\nu, S_\nu; X) = \sum_{\beta=1}^{\varrho} n_\beta(X)\,\Omega_\beta(\zeta_\nu, S_\beta; X) + n'_\beta(X)\,\Omega'_\beta(\zeta_\nu, S_\nu, X) + \\ + 2\pi i\, n''_\nu(X) \equiv S_\varrho(\zeta_\nu, S_\nu; X) \end{aligned} \tag{15}$$

$(\nu = 1, \ldots, r(X))$. Thus from (10), (13), (14) and (15)

$$\begin{aligned} \int_{\mathfrak{L}} F(\zeta, S, X)\,\frac{d\zeta}{\zeta} = \sum_{\nu=1}^{r(X)} [c'_\nu(X)\; S_\varrho(\zeta_\nu, S_\nu; X) + \overline{R}_\nu(X)] - \\ - \sum_{\alpha=1}^{\varrho} [f'_\alpha(X)\; S_{\alpha\varrho}(X) - f_\alpha(X)\; S'_{\alpha\varrho}(X)], \end{aligned} \tag{16}$$

where $c'_\nu(X) = c_\nu(X)$ if (ζ_ν, S_ν) lies inside $\mathfrak{L}$ and 0 otherwise and $\overline{R}_\nu(X)$ is an algebraic functions of X[7]. Consequently

Theorem 2.2. *An integral of the form* (7) *with an algebraic associate* $F(\zeta, S, X)$ *can be represented as a linear combination with algebraic coefficients of normal integrals of the first, second and third kinds associated with the Riemann surface* $\mathfrak{R}(X)$ *of the algebraic equation* $V(\zeta, S, X) = 0$, *increased by an algebraic function, as given by* (16).

The expression (16) is in general a transcendental function of X defined in $\mathfrak{T}^3$. ($\mathfrak{T}^3$ is a domain with infinitely many space sheets defined over the Riemann domain $\mathfrak{R}^3$.) BERGMAN [2] characterized these functions by showing that all the *period functions belonging to a given algebraic equation, considered as a function of x (or y or z), satisfy a system of ordinary differential equations with algebraic coefficients* and can be singular only if two branch points coincide, that is, only along the set $\mathfrak{S}^1$. It is possible to cut $\mathfrak{R}^3$ by a piece of a surface $\mathfrak{T}^2$ such that in the domain $\mathfrak{D}^3 = \mathfrak{R}^3 - \mathfrak{T}^2$ every closed curve can be reduced to a point. In this case the period functions $\omega_{\alpha\beta}(X)$, $\omega'_{\alpha\beta}(X)$, $\eta_{\alpha\beta}(X)$ and $\eta'_{\alpha\beta}(X)$ are single-valued in $\mathfrak{D}^3$. If, however, starting at X_0 and moving along a closed curve which intersects $\mathfrak{T}^2$ and which cannot be reduced continuously to a point without intersecting $\mathfrak{S}^1$, we return to X_0, each of the period functions will assume a new value. Similarly for $\Omega_\beta(\zeta_\nu, S_\nu; X)$. Following BERGMAN [2] we may say that the totality of functions represented by integral (7) with common algebraic associate $F(\zeta, S, X)$ *belong to the same category*.

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[7] BERGMAN called the functions (16) "period functions".

3. Expression for the integral of a harmonic vector with an algebraic associate. Let $\mathfrak{J}^1$ be a closed sufficiently smooth oriented curve lying in $\mathfrak{R}_3 - S^1$, subject to certain restrictions formulated in Paragraph 4:

$$\mathfrak{J}^1 = E[X \mid x = x(s),\ y = y(s),\ z = z(s),\ 0 \leqq s \leqq s_0], \tag{17}$$

and consider the integral

$$\begin{aligned} I_1 &= 2\pi i \int_{\mathfrak{J}^1} \vec{H}(X) \cdot \vec{dX} \\ &= \int_{\mathfrak{J}^1} dx \int_{\mathfrak{L}} F(\zeta, S, X) \frac{d\zeta}{\zeta} + \frac{1}{2} i\, dy \int_{\mathfrak{L}} (\zeta + \zeta^{-1}) F \frac{d\zeta}{\zeta} + \frac{1}{2} dz \int_{\mathfrak{L}} (\zeta - \zeta^{-1}) F \frac{d\zeta}{\zeta}. \end{aligned} \tag{18}$$

In general $\mathfrak{J}^1$ will meet the surfaces $\mathfrak{N}^2$ and $\mathfrak{B}^2$ introduced in (8) and (9) in a set of points X_k (supposed finite) $(k=0, \ldots, s)$, thus subdividing the curve $\mathfrak{J}^1$ into a finite number, s, of segments $\mathfrak{J}^1_k$, $\mathfrak{J}^1 = \sum_{k=1}^{s} \mathfrak{J}^1_k$ with end-points X_{k-1}, X_k such that for X in the interior of $\mathfrak{J}^1_k$, $\mathfrak{L}$ does not intersect any of the branch points $e_j(X)$ and the number of poles of $F(\zeta, S, X)$ inside $\mathfrak{L}$ is constant. The branch points of $R(X)$ are defined by $q_n(\zeta, X) = 0$ or $d(\zeta, X) = 0$. If a branch point also lies on $\mathfrak{L}$, then $X \in \mathfrak{B}^2$ by definition of $\mathfrak{B}^2$. Hence X is one of the end-points X_k of the segments $\mathfrak{J}^1_k$. Similarly if X is in the interior of $\mathfrak{J}^1_k$, no pole of $F(\zeta, S, X)$ meets $\mathfrak{L}$, that is, no zero $\zeta_j(X)$ of $Q(\zeta, X)$ meets $\mathfrak{L}$. Also since the zeros $\zeta_j(X)$ vary continuously as ζ moves continuously on $\mathfrak{L}$ and $\mathfrak{L}$ is a closed curve on $\mathfrak{R}(X)$, $\zeta_j(X)$ can move in or out of $\mathfrak{L}$ only for a value of X for which $\zeta_j(X)$ lies on $\mathfrak{L}$. Thus for X in the interior of $\mathfrak{J}^1_k$ there exists a sequence $T_k(\mathfrak{L})$ of the zeros of $Q(\zeta, X)$ such that $\zeta_\nu(X)$, $\nu \in T_k(\mathfrak{L})$, lie inside $\mathfrak{L}$ and the remaining zeros lie outside $\mathfrak{L}$.

Thus we consider

$$I_1 = \sum_{k=1}^{s} \sum_{\nu \in T_k(\mathfrak{L})} \Lambda^{(\nu)}(X_k, X_{k-1}) \tag{19}$$

where

$$\Lambda^{(\nu)}(X_k, X_{k-1}) = 2\pi i \int_{\mathfrak{J}^1_k} \vec{H}^{(\nu)}(X) \cdot \vec{\mathrm{d}X}. \tag{20}$$

By means of (16)

$$\begin{aligned} I_1 &= \sum_{k=1}^{s} \sum_{\nu \in T_k(\mathfrak{L})} \Lambda^{(\nu)}(X_k, X_{k-1}) \\ &= \sum_{k=1}^{s} \Bigg\{ \sum_{\nu \in T_k(\mathfrak{L})} \int_{\mathfrak{J}^1_k} [S_\varrho(\zeta_\nu, S_\nu; X)\, \vec{c}_\nu(X) \cdot \vec{dX} + \vec{\vec{R}}_\nu(X) \cdot \vec{dX}] - \\ &\quad - \sum_{\alpha=1}^{\varrho} \int_{\mathfrak{J}^1_k} [S_{\alpha\varrho}(X)\, \vec{f}'_\alpha(X) \cdot \vec{dX} - S'_{\alpha\varrho}(X)\, \vec{f}_\alpha(X) \cdot \vec{dX}] \Bigg\}, \end{aligned} \tag{21}$$

where $\vec{c}_\nu$, $\vec{f}_\alpha$, $\vec{f}'_\alpha$ and $\vec{R}_\nu$ are vectors whose first components are given by the corresponding functions in (16).

4. Relation between the functions $\Lambda(X, X_1)$ and integrals of certain transcendental functions of a complex variable. For fixed ζ the irreducible algebraic equation (2) defines an n-sheeted Riemann surface $\mathfrak{R}_1(\zeta)$ of genus ϱ_1 lying in the u-plane. The finite branch points $b_\nu(\zeta)$ of $\mathfrak{R}_1(\zeta)$ are included in the zeros of the leading coefficient $p_n(u, \zeta)$ of (2) or of the discriminant $d_1(u, \zeta)$ of (2). As in the case of the Riemann surface $\mathfrak{R}(X)$ the number of sheets is less than n if for some ζ the polynomial $p_n(u, \zeta)$ is zero for all u, which could happen for only a finite set S'_1 of points ζ. Similarly the number of branch points or their multiplicity and hence the genus of $\mathfrak{R}_1(\zeta)$ could change for only a finite set S'_2 in the ζ-plane. Assume that the curve $\mathfrak{L}$ intoduced above does not meet $S'_1 \cup S'_2$.

Interchanging the order of integration in (18) yields

$$I_2 = \int\limits_{\mathfrak{L}} \frac{d\zeta}{\zeta} \int\limits_{\mathfrak{J}(\zeta)} f(u, \zeta, S)\, du \tag{22}$$

where $du = dx + \frac{1}{2} i\, dy(\zeta + \zeta^{-1}) + \frac{1}{2} dz(\zeta - \zeta^{-1})$ and $\mathfrak{J}(\zeta)$ is the projection of $\mathfrak{J}^1$ on the u-plane:

$$\mathfrak{J}(\zeta) = E[u \,|\, u = u(X, \zeta),\ X \in \mathfrak{J}^1]. \tag{23}$$

We assume that $\mathfrak{J}^1$ in (17) can be chosen so that $\mathfrak{J}(\zeta)$ bounds a domain on $\mathfrak{R}_1(\zeta)$ for all $\zeta \in \mathfrak{L}$ except possibly a finite number. If we set $\zeta = re^{i\vartheta}$, then for $r \neq 1$, $\mathfrak{J}(\zeta)$ is the curve
$$\left(\frac{\operatorname{Re} u - x}{r - 1/r}\right)^2 + \left(\frac{\operatorname{Im} u}{r + 1/r}\right)^2 = \frac{1}{4}(y^2 + z^2).$$

Since the polynomial $p_n(u, \zeta)$ of (2) equals $q_n(X, \zeta)$ of (3) divided by a certain power ζ^q of ζ and with u replaced by $u(X, \zeta)$, $p_n[u(X, \zeta), \zeta] = 0$ and $q_n(X, \zeta) = 0$ define the same set of points $\mathfrak{B}_1^1(\zeta)$ in E_3. Also if $S_1(u, \zeta)$ is a root of (2), then $S'_1(X, \zeta) = S_1[u(X, \zeta), \zeta]$ is a root of $V(\zeta, S, X) = 0$:

$$0 = v[u, \zeta, S_1(u, \zeta)] - v[u(X, \zeta), \zeta, S_1(u(X, \zeta), \zeta)] = \zeta^{-p_0} V[\zeta, S'_1(X, \zeta), X]$$

since $v(u, \zeta, S) = v[u(X, \zeta), \zeta, S] = \zeta^{-p_0} V(\zeta, X, S)$ and hence $d(X, \zeta) = 0$ and $d_1[u(X, \zeta), \zeta] = 0$ define the same set of points $\mathfrak{B}_2^1(\zeta)$ in E_3. Let $[\zeta_j]$ be the set of points on $\mathfrak{L}$ which correspond to the set X_k in which $\mathfrak{J}'$ meets $\mathfrak{N}^2$ and $\mathfrak{B}^2$; the set $[\zeta_j]$ is defined as follows: $X_k \in \mathfrak{B}^2 \cup \mathfrak{N}^2$ implies $X_k \in \mathfrak{B}_1^1(\zeta) \cup \mathfrak{B}_2^1(\zeta) \cup \mathfrak{N}^1(\zeta)$ for some $\zeta = \zeta_j \in \mathfrak{L}$ [*cf.* (8) and (9)]. Since $\mathfrak{L}$ is an oriented curve, the ζ_j can be enumerated in the order in which we meet them moving continuously along $\mathfrak{L}$, and $\mathfrak{L}$ will be subdivided into say s' parts having end-points ζ_{j-1} and ζ_j, $\zeta_0 = \zeta_{s'}$.

According to the Weierstrass formula used in (10)

$$\begin{aligned} f(u, \zeta, X) = \sum_{\nu=1}^{r_1(\zeta)} d_\nu(\zeta)\, G_1(u_\nu, S_\nu; u, S; \zeta) + \frac{d}{du} \sum_{\nu=1}^{r_1(\zeta)} R_{1\nu}(u, S, \zeta) - \\ - \sum_{\alpha=1}^{\varrho_1} [g'_\alpha(\zeta)\, G_{1\alpha}(u, S, \zeta) - g_\alpha(\zeta)\, G'_{1\alpha}(u, S, \zeta)], \end{aligned} \tag{24}$$

where (u_ν, S_ν) are the position of the poles of $f(u_{t'}, S_{t'}, \zeta) \dfrac{du}{dt'} t'$ (t' is local uniformizing variable on $\mathfrak{R}_1(\zeta)$) and the functions d_ν, G_1, $G_{1\alpha}$, $G'_{1\alpha}$, g_α, g'_α have similar interpretations to the corresponding functions in (10). Also $R_{1\nu}$, d_ν, g_α and g'_α are algebraic functions of ζ. By a proof similar to that for formula (21) the integral I_2 of (22) can be expressed in the form

$$I_2 = \sum_{j=1}^{t} \left\{ \sum_{\nu \in T_j'(\mathfrak{J})} \int_{\mathfrak{L}_j} [d_\nu(\zeta)\, T_{\varrho_1}(u_\nu, S_\nu; \zeta) + \overline{R}_{1\nu}(\zeta)] \frac{d\zeta}{\zeta} - \right. \tag{25}$$
$$\left. - \sum_{\alpha=1}^{\varrho_1} \int_{\mathfrak{L}_j} [g'_\alpha(\zeta)\, T_{\alpha\varrho_1}(\zeta) - g_\alpha(\zeta)\, T'_{\alpha\varrho_1}(\zeta)] \frac{d\zeta}{\zeta} \right\},$$

where $\overline{R}_{1\nu}(\zeta)$, $g_\alpha(\zeta)$ and $g'_\alpha(\zeta)$ are algebraic functions of ζ; $T_{\alpha\varrho_1}(\zeta)$, $T'_{\alpha\varrho_1}(\zeta)$, $T_{\varrho_1}(u_\nu, S_\nu; \zeta)$ are transcendental functions corresponding to the S's defined by (14) and (15), namely:

$$T_{\alpha\varrho_1}(\zeta) = \sum_{\beta=1}^{\varrho_1} [2 m_\beta(\zeta)\, \sigma_{\alpha\beta}(\zeta) + 2 m'_\beta(\zeta)\, \sigma'_{\alpha\beta}(\zeta)]$$
$$T'_{\alpha\varrho_1}(\zeta) = \sum_{\beta=1}^{\varrho_1} [2 m_\beta(\zeta)\, \sigma_{\alpha\beta}(\zeta) + 2 m'_\beta(\zeta)\, \sigma'_{\alpha\beta}(\zeta)] \tag{26}$$
$$T_{\varrho_1}(u_\nu, S_\nu; \zeta) = \sum_{\beta=1}^{\varrho_1} [m_\beta(\zeta)\, \Sigma_\beta(u_\nu, S_\nu, \zeta) + m'_\beta(\zeta)\, \Sigma'_\beta(u_\nu, S_\nu, \zeta)] + 2\pi i\, m''_\nu(\zeta)$$

$(\alpha = 1, \ldots, \varrho_1;\ \nu = 1, \ldots, r_1)$; $[2\sigma_{\alpha\beta}(\zeta)]$, $[2\sigma'_{\alpha\beta}(\zeta)]$ and $[\Sigma_\beta(u_\nu, S_\nu, \zeta)]$ being systems of primitive periods for Abelian integrals of first, second and third kind respectively and $m_\beta(\zeta)$, $m'_\beta(\zeta)$, $m''_\nu(\zeta)$ integers. As in the preceding case for $\zeta \in$ interior of $\mathfrak{L}_j$, $\mathfrak{J}(\zeta)$ does not intersect any of the branch points $b_\nu(\zeta)$ and the number of poles (u_ν, S_ν) inside $\mathfrak{J}(\zeta)$ is constant. Also there exists a sequence $T_j'(\mathfrak{J}^1)$ of the roots of $q(u, \zeta) = 0$ such that the roots $u_\nu(\zeta)$, $\nu \in T_j'(\mathfrak{J}^1)$, lie inside $\mathfrak{J}(\zeta)$ and the remaining roots lie outside. The points ζ_j and X_k of (21) are connected by one of the algebraic equations $q[u(X, \zeta), \zeta] = 0$, $p_n[u(X, \zeta), \zeta] = 0$ or $d_1[u(X, \zeta), \zeta] = 0$.

Since $I_1 = I_2$ we have obtained a relation between a sum of transcendental functions $\Lambda(X_{k-1}, X_k)$ and a linear combination of certain integrals of algebraic and transcendental functions given by (25). Hence

Theorem 2.3. *Let* $\vec{H} = \vec{B}(f, \mathfrak{L}, X_0)$ *be a harmonic vector defined by* (1.2) *and* (1.6) *with an algebraic associate* $f(u, S, \zeta)$, u, ζ *and* S *being connected by the algebraic equation* (2). *In general the components of* $\vec{B}(f, \mathfrak{L}, X_0)$ *represent a finite number of infinitely multiple-valued transcendental functions. Let* $H_k^{(\nu)}(X)$ $(k = 1, 2, 3;$ $\nu = 1, 2, \ldots, n+m)$[8] *represent one of these infinitely multiple-valued functions and let* $\mathfrak{J}^1$ *be the simple closed curve given by* (17). *Then*

$$\sum_{k=1}^{s} \sum_{\nu \in T_k(\mathfrak{L})} \Lambda^{(\nu)}(X_{k-1}, X_k) = \sum_{j=1}^{t} \left\{ \sum_{\nu \in T_j'(\mathfrak{J}^1)} \int_{\zeta_{j-1}}^{\zeta_j} [d_\nu(\zeta)\, T_{\varrho_1}(u_\nu, S_\nu; \zeta) + \right. \tag{27}$$
$$\left. + \overline{R}_{1\nu}(\zeta)] \frac{d\zeta}{\zeta} - \sum_{\alpha=1}^{\varrho_1} \int_{\zeta_{j-1}}^{\zeta_j} [g'_\alpha(\zeta)\, T_{\alpha\varrho_1}(\zeta) - g_\alpha(\zeta)\, T'_{\alpha\varrho_1}(\zeta)] \frac{d\zeta}{\zeta} \right\},$$

where $\Lambda^{(\nu)}(X_{k-1}, X_k) = \int_{X_{k-1}}^{X_k} \vec{H} \cdot \vec{dX}$, $u_\nu(\zeta)$ *are the roots of* $q(u, \zeta) = 0$ *and* $\zeta_\nu(X)$ *of* $Q(X, \zeta) = 0$; $T_k(\mathfrak{L})$ *and* $T_j'(\mathfrak{J}^1)$ *are the sequences described above;* X_k *and* ζ_j *are connected by algebraic equations. The functions* $d_\nu(\zeta)$, $R_{1\nu}$, g_α *and* g'_α *are algebraic while the remaining functions on the right side of* (27) *are transcendental.*

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[8] m is the degree of $Q(X, \zeta)$ in ζ.

§ 3. Examples

1. An Example. Let

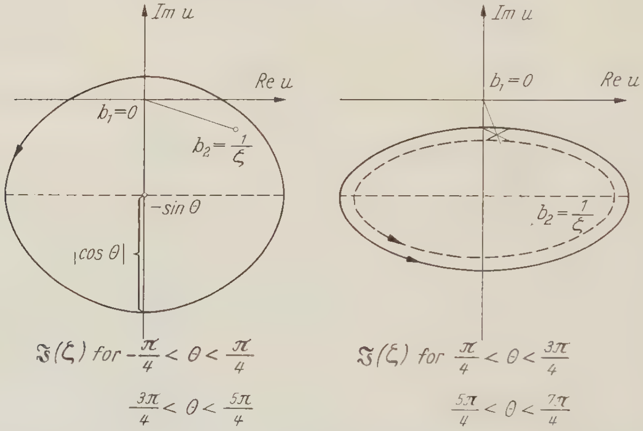
$$S^2 = u\zeta(u\zeta - 1), \qquad \zeta \neq 0. \tag{1}$$

For fixed ζ the Riemann surface $\mathfrak{R}_1(\zeta)$ of genus 0 consists of 2 sheets with branch points at $b_1(\zeta) = 0$ and $b_2(\zeta) = 1/\zeta$; also, $b_1 \neq b_2$ for any $\zeta\,(\neq \infty)$. The curves $\mathfrak{B}_1^1(\zeta)$ and $\mathfrak{B}_2^1(\zeta)$ are defined by

$$x + \tfrac{1}{2} i\, y(\zeta + \zeta^{-1}) + \tfrac{1}{2} z(\zeta - \zeta^{-1}) = 0$$

and

$$x + \tfrac{1}{2} i\, y(\zeta + \zeta^{-1}) + \tfrac{1}{2} z(\zeta - \zeta^{-1}) = 1/\zeta$$



$\mathfrak{I}(\zeta)$ for $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$
$\frac{3\pi}{4} < \theta < \frac{5\pi}{4}$

$\mathfrak{I}(\zeta)$ for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$
$\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$

Fig. 1. Projection $\mathfrak{I}(\zeta)$ of curve $\mathfrak{I}^1$ on the Riemann surface $\mathfrak{R}_1(\zeta)\,(\zeta = e^{i\vartheta})$

respectively. Let $\mathfrak{L}$ be the unit circle

$$\mathfrak{L} = E[\zeta \mid \zeta = e^{i\vartheta},\ 0 \leqq \vartheta \leqq 2\pi]. \tag{2}$$

For $\zeta \in \mathfrak{L}$, $\mathfrak{B}_1^1(\zeta)$ is the line

$$y \cos\vartheta + z \sin\vartheta = 0, \qquad x = 0$$

and

$$\mathfrak{B}_1^2 = E[X \mid x = 0]. \tag{3}$$

The equation of $\mathfrak{B}_2^1(\zeta)$ for $\zeta \in \mathfrak{L}$ is

$$y \cos\vartheta + z \sin\vartheta = -\sin\vartheta, \qquad x = \cos\vartheta$$

and

$$\mathfrak{B}_2^2 = E[X \mid x = \cos\vartheta,\ y\cos\vartheta + z\sin\vartheta = -\sin\vartheta,\ 0 \leqq \vartheta \leqq 2\pi]. \tag{4}$$

Call $\mathfrak{B}^2 = \mathfrak{B}_1^2 \cup \mathfrak{B}_2^2$.

Replacing u by $u(\zeta, X)$,

$$S^2 = (Z\zeta^2 + x\zeta + Z^*)(Z\zeta^2 + x\zeta + Z^* - 1) \qquad (X \neq 0). \tag{5}$$

In general $\mathfrak{R}(X)$ is a 2-sheeted Riemann surface of genus 1 with branch points for $Z \neq 0$ at

$$(6)\qquad \begin{aligned} e_1(X) &= \frac{-x+r}{2Z}, & e_2(X) &= \frac{-x-r}{2Z} \\ e_3(X) &= \frac{-x+[r^2+4Z]^{\frac{1}{2}}}{2Z}, & e_4(X) &= \frac{-x-[r^2+4Z]^{\frac{1}{2}}}{2Z} \end{aligned}$$

$(r^2 = x^2+y^2+z^2)$. The branch points $e_j(X)$ are coincident if and only if $r=0$, $r^2+4Z=0$ or $\pm[r^2+4Z]^{\frac{1}{2}} = \pm r$. The first and last conditions imply $X=0$ and $Z=0$ respectively and the second condition that $x^2+y^2+z^2+2z=0$ and $y=0$. Thus the set $\mathfrak{S}^1$ is given by

$$(7)\qquad \mathfrak{S}^1 = E[X \mid x^2+(z+1)^2 = 1,\ y=0] \cup E[X \mid y=z=0].$$

Let $\mathfrak{J}^1$ be the curve

$$(8)\qquad \mathfrak{J}^1 = E[X \mid x = \cos s,\ y = \sin s,\ z = -1,\ 0 \leqq s \leqq 4\pi].$$

For $\zeta \in \mathfrak{L}$

$$(9)\qquad \mathfrak{J}(\zeta) = E[u \mid \operatorname{Re} u = \cos s,\ \operatorname{Im} u = \cos\vartheta \sin s - \sin\vartheta]$$

or for $\cos\vartheta \neq 0$

$$(10)\qquad (\operatorname{Re} u)^2 + \left(\frac{\operatorname{Im} u + \sin\vartheta}{\cos\vartheta}\right)^2 = 1.$$

$\mathfrak{J}^1$ meets $\mathfrak{B}_1^2$ in the points $(0, 1, -1)$ and $(0, -1, -1)$ and $\mathfrak{B}_2^2$ in $(1, 0, -1)$ and $(-1, 0, -1)$ as well as in the two preceding points. Thus $X_1 = (1, 0, -1)$, $X_2 = (0, 1, -1)$, $X_3 = (-1, 0, -1)$, $X_4 = (0, -1, -1)$ and $\mathfrak{J}^1$ is divided into four parts with end-points at these points. (Note that X_1 and X_3 lie on $\mathfrak{S}^1$ also.) The corresponding points on $\mathfrak{L}$ are $\zeta_1 = 1$, $\zeta_2 = (1+i)/2^{\frac{1}{2}}$, $\zeta_3 = i$, $\zeta_4 = (-1+i)/2^{\frac{1}{2}}$, $\zeta_5 = -1$, $\zeta_6 = (-1-i)/2^{\frac{1}{2}}$, $\zeta_7 = -i$, $\zeta_8 = (1-i)/2^{\frac{1}{2}}$.

From (10) we see that the branch point $b_1(\zeta) = 0$ lies inside $\mathfrak{J}(\zeta)$ if $-\frac{1}{4}\pi < \vartheta < \frac{1}{4}\pi$ or $\frac{3\pi}{4} < \vartheta < \frac{5\pi}{4}$ and outside if $\frac{1}{4}\pi < \vartheta < \frac{3\pi}{4}$ or $\frac{5\pi}{4} < \vartheta < \frac{7\pi}{4}$. On the other hand $b_2(\zeta) = \cos\vartheta - i\sin\vartheta$, lies inside $\mathfrak{J}(\zeta)$ except for $\vartheta = 0$ and π.

For $X \in \mathfrak{J}^1$

$$|e_1(X)|^2 = \frac{2^{\frac{1}{2}} - \cos s}{2^{\frac{1}{2}} + \cos s}, \qquad |e_2(X)|^2 = \frac{2^{\frac{1}{2}} + \cos s}{2^{\frac{1}{2}} - \cos s}$$

so that $|e_1(X)| < 1$, $|e_2(X)| > 1$ for $-\frac{1}{2}\pi < s < \frac{1}{2}\pi$ and $|e_1(X)| > 1$, $|e_2(X)| < 1$ for $\frac{1}{2}\pi < s < \frac{3\pi}{2}$. Also for $\sin s > 0$

$$|e_3(X)|^2 = \frac{\cos^2 s - 2[\sin s]^{\frac{1}{2}}\cos s + 2\sin s}{1+\sin^2 s}$$

$$|e_4(X)|^2 = \frac{\cos^2 s + 2[\sin s]^{\frac{1}{2}}\cos s + 2\sin s}{1+\sin^2 s}.$$

It is found that for $0 < s < \frac{1}{2}\pi$, $|e_3(X)| < 1$ and $|e_4(X)| > 1$ and for $\frac{1}{2}\pi < s < \pi$, $|e_3(X)| > 1$ and $|e_4(X)| < 1$. For $\sin s < 0$, the values of $|e_3|^2$ and $|e_4|^2$ are equal to the values of $|e_4|^2$ and $|e_3|^2$ respectively when $\sin s > 0$.

2. Example, Continued. Now consider the rational function

$$(11)\qquad f(u, \zeta, S) = \frac{p(u, \zeta, S)}{q(u, \zeta)}$$

where $u_\nu(\zeta)$ are the zeros of $q(u, \zeta)$. If we assume that the surface $\mathfrak{R}^2$ lying in E_3 never intersects $\mathfrak{J}^1$, then the zeros of $Q(\zeta, X)$ do not lie on $\mathfrak{L}$ for $X \in \mathfrak{J}^1$ or the zeros of $q(u, \zeta)$ on $\mathfrak{J}(\zeta)$. Proof: If for some $X_0 \in \mathfrak{J}^1$, there exists a $\zeta_0 \in \mathfrak{L}$ such that $Q(\zeta_0, X_0) = 0$, then $X_0 \in \mathfrak{R}^2$ by definition of $\mathfrak{R}^2$. [See (2.8)]. In the second case $X_0 \in \mathfrak{J}^1$ implies $u = u(\zeta, X_0) \in \mathfrak{J}(\zeta)$ for all ζ. If a zero of $q(u, \zeta) = q[u(\zeta, X), \zeta] = \zeta^{-p} Q(\zeta, X)$ lies on $\mathfrak{J}(\zeta)$ for some $\zeta = \zeta_0 \in \mathfrak{L}$, then $u(\zeta_0, X_0) \in \mathfrak{J}(\zeta_0)$. But $0 = q(u, \zeta_0) = q[u(\zeta_0, X_0), \zeta_0] = \zeta_0^{-p} Q(\zeta_0, X)$. Therefore $X_0 \in \mathfrak{R}^2$ which is a contradiction. Thus $\mathfrak{R}^2 \cap \mathfrak{J}^1 = 0$ implies the conclusion stated above. Consequently in considering the integrals $\int\limits_{\mathfrak{J}^1} \int\limits_{\mathfrak{L}}$ and $\int\limits_{\mathfrak{L}} \int\limits_{\mathfrak{J}(\zeta)}$ we need only subdivide $\mathfrak{J}^1$ at the points $X_k (k = 1, \ldots, 4)$ and $\mathfrak{L}$ at ζ_j $(j = 1, \ldots, 8)$.

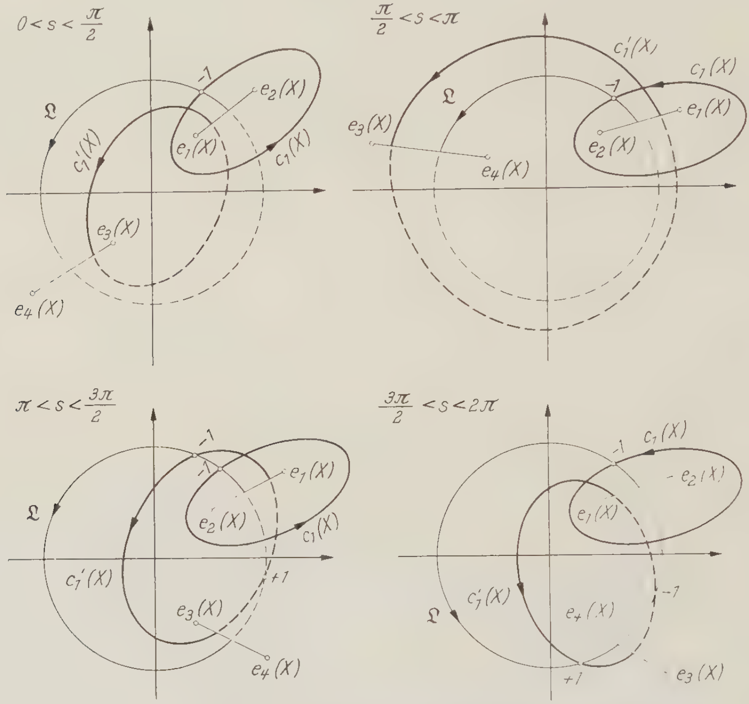


Fig. 2. Curve $\mathfrak{L}$ on Riemann surface $\mathfrak{R}(X)$ for different values of s $(X = X(s))$

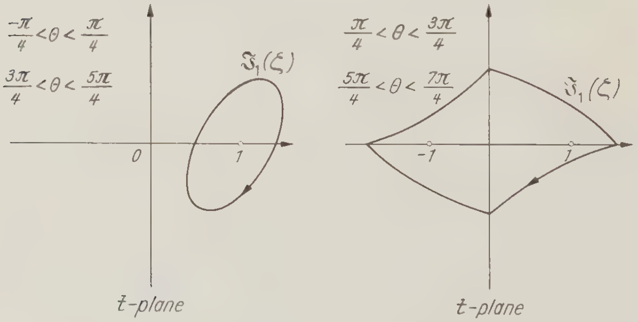


Fig. 3. Transform of $\mathfrak{J}(\zeta)$

$\mathfrak{R}(X)$ is made into a simply-connected surface by introducing a cut $c_1(X)$ and a conjugate cut $c_1'(X)$ as shown in Figure 2. Then $n_1(X) = n(X) = -1$ and $n_1'(X) = 0$ and since $\alpha = 1$, $\varrho = 1$ in (2.14)

$$\begin{aligned} S_{11}(X) &= S(X) = \pm 2\omega(X) \qquad \big(\omega(X) = \omega_{11}(X)\big), \\ S'_{11}(X) &= S'(X) = +2\eta(X) \qquad (\eta = \eta_{11}). \end{aligned}$$

Also if the loops $l_\nu(X)$ around the poles of $F(\zeta, S, X)$ can be drawn so that $\mathfrak{L}$ never intersects them, then by (2.15)

$$S_1(\zeta_\nu, S_\nu; X) = +\Omega(\zeta_\nu, S_\nu, X) \qquad (\Omega_1 = \Omega).$$

Consequently (2.16) becomes

$$\begin{aligned} \int_{\mathfrak{L}} F(\zeta, S, X) \frac{d\zeta}{\zeta} &= + \sum_{\nu=1}^{r} c_\nu'(X)\,\Omega(\zeta_\nu, S_\nu; X) + \overline{R}_\nu(X) - \\ &\quad - 2f'(X)\,\omega(X) + 2f(X)\,\eta(X) \qquad (f_\alpha = f,\ f_\alpha' = f') \end{aligned} \tag{12}$$

and by (2.21)

$$\begin{aligned} I_1 = \sum_{k=1}^{4} \int_{\mathfrak{J}_k^1} \Big\{ \sum_{\nu-1}^{r} &[\Omega(\zeta_\nu, S_\nu, X)\,\vec{c}_\nu(X) \cdot d\vec{X} + \vec{R}_\nu(X) \cdot d\vec{X}] - \\ &- 2\omega(X)\,\vec{f}'(X) \cdot d\vec{X} + 2\eta(X)\,\vec{f}(X) \cdot d\vec{X} \Big\}. \end{aligned} \tag{13}$$

Since $\mathfrak{R}_1(\zeta)$ has genus 0 it can be mapped on to the complex t-plane by the transformation

$$t = \left(\frac{u\zeta}{u\zeta - 1}\right)^{\frac{1}{2}}, \qquad u = u(t^2) = \frac{t^2}{\zeta(t^2-1)}, \tag{14}$$

where $du/dt = -2t\,dt/\zeta(t^2-1)^2$, $S \to S(t) = t/(t^2-1)$, $b_1(\zeta) = 0 \to t = 0$, $b_2(\zeta) = 1/\zeta \to \infty$ and $t^2 = 1 \to u = \infty$; $\mathfrak{J}(\zeta) \to$ a closed curve $\mathfrak{J}_1(\zeta)$ lying on the t-plane and symmetric with respect to the origin if $t=0$ lies inside the curve. If $b_1 = 0$ and $b_2 = 1/\zeta$ lie inside $\mathfrak{J}(\zeta)$, then $t_1 = 0$ and $t_2 = \infty$ lie outside $\mathfrak{J}_1(\zeta)$. If $b_1 = 0$ lies outside and $b_2 = 1/\zeta$ inside $\mathfrak{J}(\zeta)$, then $t = 0$ lies inside and $t = \infty$ outside $\mathfrak{J}_1(\zeta)$. In both cases $t = 1$ lies inside $\mathfrak{J}_1(\zeta)$. See Figure 3. Under transformation (14) $f(u, \zeta, S) \to f[u(t^2), \zeta, t/(t^2-1)] = f_1(t, \zeta)$, a rational functions of t and ζ and

$$\int_{\mathfrak{J}(\zeta)} f(u, \zeta, S)\,du = \int_{\mathfrak{J}_1(\zeta)} f_1(t, \zeta) \frac{du}{dt}\,dt = \sum_{\nu=1}^{r_1'} r_\nu(\zeta), \tag{15}$$

where $\sum r_\nu(\zeta) = 2\pi i$ times the sum of the residues of $f_1(t, \zeta)\,du/dt$ in $\mathfrak{J}_1(\zeta)$, u is the uniformizing variable given by (14). Thus finally

$$I_2 = \int_{\mathfrak{L}} \frac{d\zeta}{\zeta} \int_{\mathfrak{J}(\zeta)} f(u, \zeta, S)\,du = \sum_{j=1}^{8} \sum_{\nu=1}^{r_1'} \int_{\mathfrak{L}_j} r_\nu(\zeta) \frac{d\zeta}{\zeta}. \tag{16}$$

Equating (13) and (16) gives the final result.

3. Singularities of the period functions. According to BERGMAN [2] the period functions are singular only along algebraic curves where the roots $r_1(X)$, $r_2(X)$, $r_3(X)$ of the resolvent cubic of the fourth-degree equation (5) coincide. However r_j are connected with the branch points e_j of the Riemann surface of equation

(5) by the relations

$$r_1 = e_1 e_3 + e_2 e_4, \quad r_2 = e_1 e_4 + e_2 e_3, \quad r_3 = e_1 e_2 + e_3 e_4.$$

Hence $\quad r_j = r_k, \quad r_j \neq r_i \quad (i, j, k = 1, 2, 3;\ i, j, k \text{ distinct})$

imply $\quad e_j = e_k, \quad e_j \neq e_i \quad (i, j, k = 1, 2, 3, 4;\ i, j, k \text{ distinct}).$

These latter relations hold if and only if $X \in \mathfrak{S}^1$. Thus the period functions are singular along the circle (7).

The period functions are singular along the intersection of quadric surfaces in the following case:

Theorem 3.1. *If*

$$S^n = \prod_{j=1}^{m} [u\zeta - p_j(\zeta)] \qquad (n \geqq 2,\ m \geqq 2), \tag{17}$$

where $p_j(\zeta)$ is a quadratic function of ζ $(j = 1, \ldots, m)$, then the period functions of the Riemann surface $\mathfrak{R}(X)$ are singular along the intersection of quadric surfaces.

Proof. Without loss of generality we may consider the case

$$u\zeta - p_j(\zeta) = \zeta^2(Z - a) + (x - b)\zeta + Z^* - c$$
$$u\zeta = \zeta^2 Z + x\zeta + Z^*,$$

for otherwise we may make a translation of the coordinates x, y, z so that one expression reduces to $u\zeta$ and the degree of the intersecting surfaces will not be changed by this procedure. Branch points occur at the points

$$\frac{-(x-b) \pm r_1^{\frac{1}{2}}}{2(Z-a)}, \qquad \frac{-x \pm r_2^{\frac{1}{2}}}{2Z},$$

where $r_1 = (x-b)^2 - 4(Z-a)(Z^*-a)$ and $r_2 = x^2 - 4ZZ^*$. The branch points will be coincident for $r_1 = 0$ or $r_2 = 0$ which lead to circles or in case

$$\frac{-(x-b) \pm r_1^{\frac{1}{2}}}{2(Z-a)} = \frac{-x \pm r_2^{\frac{1}{2}}}{2Z}$$

or

$$\pm Z r_1^{\frac{1}{2}} = a x - b Z \pm (Z - a) r_2^{\frac{1}{2}}.$$

Squaring and simplifying we get

$$2(Z - a) x^2 a + 2b x Z (a - Z) - 4Z(Z - a)(a Z^* - Z c) = \pm 2(a x - b Z)(Z - a) r_2^{\frac{1}{2}}.$$

Divide by $2(Z - a)$, square and divide by $4Z^2$. After some simplification we get

$$a c x^2 - a b x Z^* - b c x Z + (b^2 - 2ac) Z Z^* + a^2 Z^{*2} + c^2 Z^2 = 0,$$

which gives in general the intersection of two quadric surfaces as the curves along which the period functions are singular. This completes the proof of Theorem 3.1.

4. Singularity curves of higher degree. However, the curves along which period functions are singular can be of higher than second degree, as the following example shows. Consider the irreducible equation

$$S^2 = u\zeta^2 - 1, \tag{18}$$

which becomes on substituting $u(X, \zeta)$ for u

$$S^2 = Z\zeta^3 + x\zeta^2 + Z^*\zeta - 1 \qquad (Z \neq 0), \tag{19}$$

and which represents a 2-sheeted Riemann surface $\mathfrak{R}(X)$ of genus 1. The period functions have singularities where the branch points coincide, that is, when the discriminant of equation (19) is zero. This leads after some simplification to the equation

$$4(3ZZ^* - x^2)^3 + x^2(2x^2 - 9ZZ^*)^2 - 54xZ^2(2x^2 - 9ZZ^*) + 729Z^4 = 0$$

or upon separating into real and imaginary parts to the equations

$$4(3ZZ^* - x^2)^3 + x^2(2x^2 - 9ZZ^*)^2 - 54x\,\mathrm{Re}\,Z^2 - (2x^2 - 9ZZ^*) + 729\,\mathrm{Re}\,Z^4 = 0$$

and

$$-2x\,\mathrm{Im}\,Z^2(2x^2 - 9ZZ^*) + 27\,\mathrm{Im}\,Z^4 = 0.$$

After some calculation these two equations are reduced to

$$(y^2 + z^2)^3 + x^2(y^2 + z^2)^2 + 2x(z^2 - y^2)(8x^2 + 9y^2 + 9z^2) - 27(z^4 - 6y^2z^2 + y^4) = 0$$

$$yz[x(8x^2 + 9y^2 + 9z^2) - 27(z - y)] = 0$$

so that the curves of intersection take the form

(a) $z^4 + x^2z^2 + 2x(8x^2 + 9z^2) - 27z^2 = 0, \qquad y = 0,$

(b) $y^4 + x^2y^2 - 2x(8x^2 + 9y^2) - 27y^2 = 0, \qquad z = 0,$

(c) $(y^2 + z^2)^3 + x^2(y^2 + z^2)^2 - 54(z - y)^2(z + y) - 27(z^4 - 6y^2z^2 + y^4) = 0,$

$x(8x^2 + 9y^2 + 9z^2) = -27(z - y),$

that is, two fourth degree curves and the intersection of surfaces represented by a sixth and a third degree equation.

§ 4. Generalization of integral theorem to vectors whose components satisfy certain linear partial differential equations

1. Line integrals of solutions of partial differential equation (1). Consider the linear partial differential equation

$$\Delta\psi + F(r^2)\psi = 0, \tag{1}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ and F is an entire function of $r^2 = x^2 + y^2 + z^2$. Bergman has shown [*4, 5*] that there exists an entire function $\Gamma(r, t)$ of r^2 for $|t| \leqq 1$:

$$\Gamma(r, t) = 1 + \sum_{n=1}^{\infty} t^{2n} b^{(n)}(r^2) \tag{2}$$

such that the operator

$$\psi(X) = P_3[H(X)] = \int_{-1}^{1} \Gamma(r, t)\, H[X(1 - t^2)]\, dt = 2\int_{0}^{1} \Gamma(r, t)\, H[X(1 - t^2)]\, dt \tag{3}$$

generates solutions of (1) from harmonic functions $H(X)$.

Let $\vec{H}(X)$ be a harmonic vector with components $H_j(X)$ $(j=1, 2, 3)$ given by (1.2) and (1.6) and consider the line integral

$$I = \int_{\mathfrak{J}^1} \vec{\psi}(X) \cdot \vec{dX} = 2 \int_{\mathfrak{J}^1} \int_0^1 \Gamma(r, t)\, dt\, \vec{H}[X(1-t^2)] \cdot \vec{dX}. \tag{4}$$

If $\mathfrak{J}^1$ is a curve lying on the sphere $x^2+y^2+z^2=r^2$, then $\Gamma(r, t)$ is independent of X and the operator $P_3[H(X)]$ represents a harmonic function $L(X)$; also, the order of integration may be changed on the right side of (4) giving

$$I = \int_{\mathfrak{J}^1} \vec{L}(X) \cdot \vec{dX} = 2 \int_0^1 \Gamma(r, t)\, dt \int_{\mathfrak{J}^1} \vec{H}[X(1-t^2)] \cdot \vec{dX}. \tag{5}$$

2. Behavior in the large of solutions of partial differential equation (1). In order to simplify matters we assume that the B_3-associate of $\vec{H}(X)$ is a rational function of u and ζ:

$$f(u, \zeta) = \frac{p(u, \zeta)}{q(u, \zeta)}, \tag{6}$$

p and q polynomials in u and ζ. Under the transformation (1.3) $f(u, \zeta)$ goes into $P(\zeta, X)/Q(\zeta, X)$, P and Q polynomials in ζ and X with Q and q connected as in (2.6). Thus

$$H[(1-t^2)X] = \int_{\mathfrak{L}} \frac{P[\zeta, (1-t^2)X]}{Q[\zeta, (1-t^2)X]} \frac{d\zeta}{\zeta}, \tag{7}$$

where $\mathfrak{L}$ is a closed regular curve in the ζ-plane.

Following BERGMAN [5] we introduce an R-manifold $\mathfrak{S}^3$ in E_3 in the following way. For fixed X the equation

$$B(t, \zeta; X) = \frac{Q[\zeta, (1-t^2)X]}{q_n(t, X)} = \sum_{\nu=0}^{n} \frac{q_\nu(t, X)}{q_n(t, X)} \zeta^\nu = 0 \tag{8}$$

$(q_\nu(t, X)$ polynomials$)$ defines a Riemann surface $\mathfrak{T}(X)$ with n sheets $\zeta=\zeta^{(\nu)}(t, X)$, $\nu=1, \ldots, n$, which in general has s distinct branch points $t^{(\mu)}(X)$, namely, the roots of the discriminant

$$D(t, X) = 0. \tag{9}$$

Equation (9) defines in E_3 an R-manifold $\mathfrak{S}^3$ with s space-sheets $t^{(\mu)}(X)$, $\mu=1, \ldots, s$. The branch line of $\mathfrak{S}^3$ is given by

$$\mathfrak{S}^1 = E[D(t, X) = 0,\ D_t(t, X) = 0]. \tag{10}$$

BERGMAN [5] has shown that if $\mathfrak{P}^1$ is a path on $\mathfrak{S}^3 - \mathfrak{S}^1$, then the genus $\varrho = \varrho(X)$ of $\mathfrak{T}(X)$ remains constant as X moves along $\mathfrak{P}^1$.

By the residue theorem

$$\int_{\mathfrak{L}} \frac{P[\zeta, (1-t^2)X]}{Q[\zeta, (1-t^2)X]} \frac{d\zeta}{\zeta} = 2\pi i \sum \quad \text{residues at poles inside } \mathfrak{L},$$

where the poles of the integrand are given by the zeros $\zeta=\zeta_\mu(t, X)$ of $Q[\zeta, (1-t^2)X]$. The residue $R(\zeta_\mu, t, X)$ at the pole ζ_μ is a rational function of ζ_μ, t and X or an algebraic function, $\frac{1}{2\pi i} R_\mu(t, X)$, of t and X. Also for fixed t and X a certain subsequence $T(\mathfrak{L}, t, X)$ of the poles lie inside $\mathfrak{L}$ and the rest outside. Thus

$$\int_{\mathfrak{L}} \frac{P[\zeta, (1-t^2)X]}{Q[\zeta, (1-t^2)X]}\,\frac{d\zeta}{\zeta} = \sum_{\mu\in T(\mathfrak{L}, t, X)} R_\mu(t, X), \tag{11}$$

where the right side is the sum of certain branches of an algebraic function $\{R_\mu(t, X)\}$ which is defined on the Riemann surface $\mathfrak{T}(X)$. For fixed X as we have seen in § 2 the interval $(0, 1)$ can be subdivided into a finite number of pieces $\tau_k(X) = \big(t_{k-1}(X), t_k(X)\big)$, $k=1, \ldots, p(X)$ such that for $t\in\tau_k(X)$ a fixed set, $T_k(\mathfrak{L}, X)$, of the zeros of $Q[\zeta, (1-t^2)X]$ lie inside $\mathfrak{L}$ and also for $t\in$ interior of $\tau_k(X)$ no $\zeta_\mu(X)$ can lie on $\mathfrak{L}$. Call

$$\Gamma(r, t) \sum_{\mu\in T_k(\mathfrak{L}, X)} R_\mu(t, X) = R^{(k)}(t, X). \tag{12}$$

By the Weierstrass formula for $X\in\mathfrak{S}^3 - \mathfrak{S}^1$,

$$\begin{aligned} R^{(k)}(t, X) = {}& \sum_{\varkappa=1}^{p_k(X)} c_{k\varkappa}^{(k)}(X)\, G(t_{k\varkappa}, \zeta_{k\varkappa}; t, \zeta; X) + \frac{d}{dt} \sum_{\varkappa-1}^{p_k(X)} R_{k\varkappa}^{(k)}(t, \zeta, X) - \\ & - \sum_{\alpha\ -1}^{\varrho} [f'_{k\alpha}(X)\, G_\alpha(t, \zeta, X) - f_{k\alpha}(X)\, G'_\alpha(t, \zeta, X)], \end{aligned} \tag{13}$$

where $(t_{k\varkappa}, \zeta_{k\varkappa})$ are the position of the poles of $R^{(k)}(t, X)$ on the Riemann surface $\mathfrak{T}(X)$, $c_\varkappa^{(k)}(X)$ is the residue of $R^{(k)}(t, X)$ at $(t_\varkappa, \zeta_\varkappa)$ and G_α, G'_α, G are Weierstrass integrands of first, second and third kind respectively. If we integrate G_α, G'_α and G from t_{k-1} to t_k we obtain

$$\begin{aligned} &\int_{t_{k-1}}^{t_k} G_\alpha(t, \zeta, X)\, dt = \bar{I}_\alpha(t_{k-1}, t_k; X) + S_{\alpha\varrho}^{(k)}(X) \\ &\int_{t_{k-1}}^{t_k} G'_\alpha(t, \zeta, X)\, dt = \bar{I}'_\alpha(t_{k-1}, t_k; X) + S_{\alpha\varrho}^{(k)\prime}(X) \\ &\int_{t_{k-1}}^{t_k} G(t_{k\varkappa}, \zeta_{k\varkappa}; t, \zeta; X)\, dt = \bar{I}(t_{k-1}, t_k; t_{k\varkappa}, \zeta_{k\varkappa}; X) + S_\varrho^{(k)}(t_{k\varkappa}, \zeta_{k\varkappa}; X) \end{aligned} \tag{14}$$

where $\bar{I}_\alpha$, $\bar{I}'_\alpha$ are the values of the integrals of G_α and G'_α respectively over a path from t_{k-1} to t_k on the simply connected surface $\mathfrak{T}'(X)$, which does not meet any of the cuts or conjugate cuts used in constructing $\mathfrak{T}'(X)$ and $\bar{I}$ is the value of the integral of G over a path from t_{k-1} to t_k which does not meet any of these cuts or the loops introduced around the poles ζ_μ; $S_{\alpha\varrho}^{(k)}(X)$, $S_{\alpha\varrho}^{(k)\prime}(X)$, $S_\varrho^{(k)}(t_{k\varkappa}, \zeta_{k\varkappa}; X)$ are linear combinations with integral coefficients of the system of primitive periods of the first, second and third kind respectively, entirely analogous to (2.14) and (2.15). By means of (14) we obtain an expression for

$$\int_{t_{k-1}(X)}^{t_k(X)} R^{(k)}(t, X)\, dt,$$

and summing $k=1$ to $p(X)$ we get that

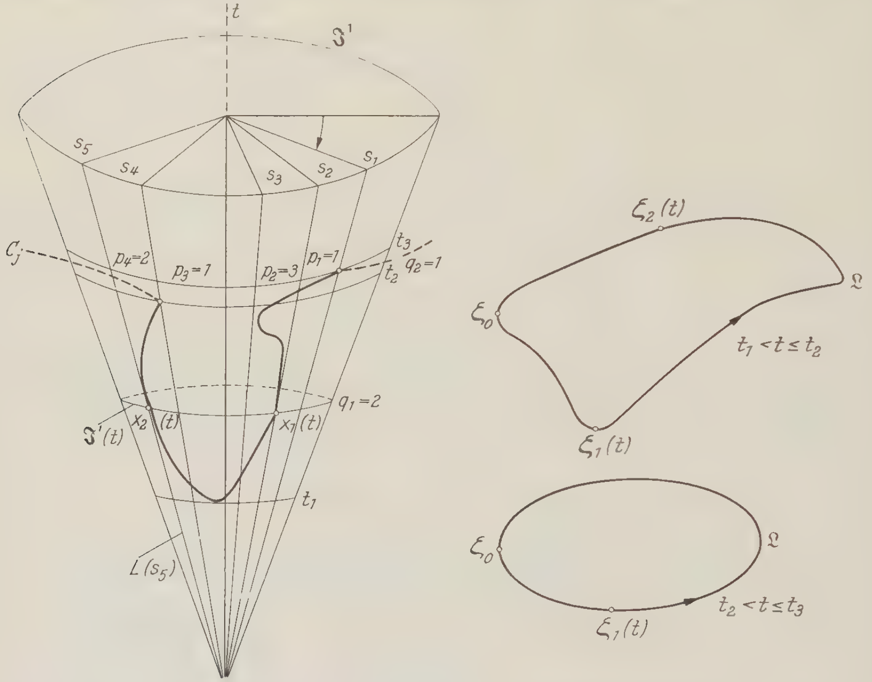
$$\sum_{k=1}^{p(X)} \int_{t_{k-1}(X)}^{t_k(X)} R^{(k)}(t, X)\, dt \tag{15}$$

is a linear sum with algebraic coefficients of

i. the transcendental functions $S^{(k)}_{\alpha\varrho}(X)$, $S^{(k)\prime}_{\alpha\varrho}(X)$, $S^{(k)}_{\varrho}(t_{k_\mu}, \zeta_{k_\mu}; X)$, which involve the period functions of the R-manifold $\mathfrak{S}^3$,

ii. integrals of algebraic functions, not involving period functions, and hence representing algebraic or logarithmic functions.

Following BERGMAN [5] we may say that a function which can be represented linearly as the sum of functions of types i and ii belongs to class $G(\mathfrak{S})$. Consequently *a solution* ψ *of* (1) *which can be represented in the form* $P_3[H(X)]$ *where* H *is a harmonic function with a rational* B_3*-associate is a function of class* $G(\mathfrak{S})$.



Surface $\mathfrak{J}^2$

Subdivision of curve $\mathfrak{L}$

Fig. 4

3. *Expression for the line integral* (4). Let $\mathfrak{J}^1$ be a closed curve lying on the sphere $x^2+y^2+z^2=r^2$ $(0<r<\infty)$ which is defined by (2.17), $\mathfrak{J}^2$ the surface in E_3 given by

(16) $$\mathfrak{J}^2 = E[X | X = t^2 X(s),\ X(s) \in \mathfrak{J},\ 0 \leqq t \leqq 1],$$

and $\mathfrak{N}^2$ that given by (2.8), namely,

(17) $$\mathfrak{N}^2 = E[X | X \in \sum_{\nu=1}^{n} \bigcup_{\zeta \in \mathfrak{L}} \mathfrak{N}^1_\nu(\zeta),$$

where

(18) $$\mathfrak{N}^1_\nu(\zeta) = E[X | u(X, \zeta) = u_\nu(\zeta)],$$

$u_\nu(\zeta)$ being the roots of $q(u, \zeta) = 0$. For fixed ζ, $\mathfrak{N}^1_\nu(\zeta)$ is a straight line in E_3.

In general $\mathfrak{J}^2$ and $\mathfrak{N}^2$ will intersect in segments of a finite number of curves $\mathfrak{C}_1^1, \ldots, \mathfrak{C}_m^1$. The equation of each $\mathfrak{C}_j^1$ may be obtained in terms of the parameter s by substituting (2.17) for X into $u(X, \zeta) = u_\nu(\zeta)$, separating the latter equation into real and imaginary parts and eliminating ζ, thereby obtaining a relation between t and s. Solving for t in terms of s and substituting back into (2.17) the desired equation is obtained for $\mathfrak{C}_j^1$. In general $\mathfrak{C}_j^1$ will be defined for s in a finite number of closed intervals of s and possibly at a set (assumed finite) of isolated points. Let $\mathfrak{L}'(s)$ be the line on $\mathfrak{J}^2$ defined by $X = t^2 X(s)$, $0 \leqq t \leqq 1$. The curve $\mathfrak{C}_j^1$ may coincide with some $\mathfrak{L}'(s_0)$ for t in an interval, in which case s_0 will be one of the points of subdivision of s. Otherwise suppose that $\mathfrak{C}_j^1$ intersects each $\mathfrak{L}'(s)$ a finite number of times. Since $\mathfrak{C}_j^1$ is a curve and $\mathfrak{L}'(s)$ varies continuously with s, the number of points of intersection of $\mathfrak{C}_j^1$ and $\mathfrak{L}'(s)$ will be constant for s in non-overlapping intervals with jumps at the end-points of the intervals. Thus s may be subdivided into intervals in which $\mathfrak{C}_j^1$ intersects each $\mathfrak{L}'(s)$ a constant number of times. Repeating this procedure for $\mathfrak{C}_k^1$ $(k \neq j)$ we obtain a subdivision of s: $0 < s_1 < \cdots < s_p$ such that for $s \in S_\varkappa = (s_{\varkappa-1}, s_\varkappa)$ there are $p_\varkappa$ points of intersection of $\{\mathfrak{C}_j^1\}$ and $\mathfrak{L}'(s)$. Let $\mathfrak{J}_\varkappa^1$ be that part of $\mathfrak{J}^1$ for which $s \in S_\varkappa$. Then for $X \in \mathfrak{J}_\varkappa^1$ (15) becomes

$$I_\varkappa = \sum_{k=1}^{p_\varkappa} \int_{t_{k-1}(X)}^{t_k(X)} R^{(k)}(t, X)\, dt. \tag{19}$$

Consequently the line integral $\int \vec{\psi}(X) \cdot \overrightarrow{dX}$ is broken up into:

$$I = \sum_{\varkappa=1}^{p} \sum_{\varkappa=1}^{p_\varkappa} \sum_{\mu \in T_{\varkappa k}(\mathfrak{L})} \int_{\mathfrak{J}_\varkappa^1} \vec{\psi}_k^{(\mu)} \cdot \overrightarrow{dX}, \tag{20}$$

where $\vec{\psi}_k^{(\mu)}$ is a vector whose components are functions of class $G(\mathfrak{S})$ and

$$\psi_k^{(\mu)}(X) = \int_{t_{k-1}(X)}^{t_k(X)} \Gamma(r, t)\, R_\mu(t, X)\, dt. \tag{21}$$

4. Relation between the line integral (4) *and certain functions of one complex variable.* Substituting (7) into (5) and interchanging the order of integration in the last two integrals, we get

$$\begin{aligned} I' &= 2 \int_0^1 \Gamma(r, t)\, dt \int_{\mathfrak{L}} \frac{d\zeta}{\zeta} \int_{\mathfrak{J}(\zeta)} \frac{p[(1-t^2)\, u, \zeta]}{q[(1-t^2)\, u, \zeta]}\, du \\ &= 2 \int_0^1 \frac{\Gamma(r, t)}{1-t^2}\, dt \int_{\mathfrak{L}} \frac{d\zeta}{\zeta} \int_{\mathfrak{J}'(\zeta, t)} \frac{p(v, \zeta)}{q(v, \zeta)}\, dv, \end{aligned} \tag{22}$$

where $\mathfrak{J}(\zeta)$ is given by (2.23) and

$$\mathfrak{J}'(\zeta, t) = E\,[v \,|\, v = u(X, \zeta)\,(1 - t^2),\ X \in \mathfrak{J}^1]. \tag{23}$$

We assume that $\mathfrak{J}'(\zeta, t)$ is a closed curve except possibly for a finite number of values of ζ. By the residue theorem

$$\int_{\mathfrak{J}'(\zeta, t)} \frac{p(v, \zeta)}{q(v, \zeta)}\, dv = \sum_\nu r_\nu [v_\nu(\zeta), \zeta], \tag{24}$$

where r_ν is $2\pi i$ times the residue of $f(v,\zeta)$ at the poles $v_\nu(\zeta)$ which lie inside $\mathfrak{J}'(\zeta,t)$. As in § 2 for each t we can subdivide $\mathfrak{L}$ into a finite number of pieces $\mathfrak{L}_j(t)$ with end-points $\zeta_{j-1}(t)$, $\zeta_j(t)$ in each of which the set of zeros of $q(v,\zeta)$ is constant. Thus

$$2\int\limits_0^1 \frac{\Gamma(r,t)}{1-t^2}\,dt \sum_{j=1}^{q(t)} \sum_{\nu\in T_j'(t,\mathfrak{L})} \int\limits_{\mathfrak{L}_j(t)} r_\nu[v_\nu(\zeta),\zeta]\,\frac{d\zeta}{\zeta}\,. \tag{25}$$

In the considerations of Paragraph 3 we could have subdivided $(0\leqq t\leqq 1)$ into a finite number of pieces $\tau_\alpha=(t_{\alpha-1},t_\alpha)$ and $\mathfrak{J}'(t)$ into a finite number of pieces $\mathfrak{J}_j'(t)=\big(X_{j-1}(t),X_j(t)\big)$ such that for $t\in\tau_\alpha$ and $X\in\mathfrak{J}_j'(t)$ the number of points of intersection of $\mathfrak{J}^2$ and the curves $\{\mathfrak{C}_j^1\}$ is a constant $\big(\mathfrak{J}'(t)$ is the intersection of $\mathfrak{J}^2$ with the plane $t=t\big)$. For each $t\in\tau_\alpha$, $X_j(t)\in\mathfrak{N}^2$ implies that there exists a ν and a $\zeta(t)\in\mathfrak{L}$ such that $X_j(t)\in\mathfrak{N}_\nu^1\big(\zeta(t)\big)$. We assume that $X=X_j(t)$ intersects each $\mathfrak{N}_\nu^1(\zeta)$ either in a finite number of points or coincides with $\mathfrak{N}_\nu^1(\zeta)$ for t in a finite number of subintervals of τ_α, which case gives one of the points of subdivision of $\mathfrak{L}$. In the former case since $X_j(t)$ moves continuously with t and $\mathfrak{N}_\nu^1(\zeta)$ is a straight line, τ_α can be subdivided further into a finite number of non-overlapping subintervals on each of which the number of points of intersection of $\{\mathfrak{C}_j^1\}$ and $\mathfrak{N}_\nu^1(\zeta)$ is constant with a jump at the end-points of the τ-intervals. Reletter the subdivisions into a simple sequence τ_ι $(\iota=1,\ldots,q)$ such that for $t\in\tau_\iota$, q_ι of the points $X_\iota(t)$ lie on $\mathfrak{N}_\nu^1(\zeta)$, say $\zeta_1(t),\ldots,\zeta_{q_\iota}(t)$. These points also lie on $\mathfrak{L}$ and hence subdivide it into q_ι pieces $\mathfrak{L}_i$ with end-points $\zeta_{i-1}(t)$, $\zeta_i(t)$. If the $v_\nu(\zeta)$ are the roots of $q(v,\zeta)=0$, as in the case in Paragraph 3 there is a sequence $T_{\iota i}'(\mathfrak{J})$ of the $v_\nu(\zeta)$ which lie inside the domain bounded by $\mathfrak{J}'(t,\zeta)$ while the remaining $v_\nu(\zeta)$ lie outside. Thus (25) becomes

$$2\sum_{\iota=1}^{q}\sum_{i=1}^{q_\iota}\sum_{\nu\in T_{i\iota}'(\mathfrak{J}^1)} \int\limits_{t_{\iota-1}}^{t_\iota} \frac{\Gamma(r,t)}{1-t^2}\,dt \int\limits_{\mathfrak{L}_i(t)} r_\nu[v_\nu(\zeta),\zeta]\,\frac{d\zeta}{\zeta}\,. \tag{26}$$

Equating I in (20) and (26) gives us the following

Theorem 4.1. *Let $\vec{\psi}$ be a vector whose components are solutions of* (1) *which can be represented in the form $P_3[H(X)]$ where H is a harmonic function with a rational B_3-associate p/q. Let $\mathfrak{S}^3$ be the R-manifold on which $\vec{\psi}$ is single-valued. Let $\mathfrak{J}^1$ be a simple closed curve situated on the sphere $E[x^2+y^2+z^2=r^2,\ 0< \ <\infty]$, satisfying conditions noted above. Then*

$$\sum_{\varkappa=1}^{p}\sum_{k=1}^{p_\varkappa}\sum_{\mu\in T_{\varkappa k}(\mathfrak{L})} \int\limits_{X_{k-1}}^{X_k} \vec{\psi}_k^{(\mu)}\cdot d\vec{X} \tag{27}$$

is equal to (26), *where $v_\nu(\zeta)$ are the roots of $q(v,\zeta)=0$ and $T_{\varkappa k}(\mathfrak{L})$ and $T_{\iota i}'(\mathfrak{J}^1)$ are the sequences described above.*

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Quadratic transformations of generalized Legendre's associated functions for special values of the parameters

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1. Introduction

In previous papers [*1*], [*2*] KUIPERS and the author gave two linearly independent solutions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ of the differential equation:

$$(1) \qquad (1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left\{k(k+1) - \frac{m^2}{2(1-z)} - \frac{n^2}{2(1+z)}\right\} w = 0,$$

specified either in the form of integrals over double circuits, or in terms of hypergeometric functions valid in restricted areas of the z-plane in which a cross-cut exists along the real axis from 1 to $-\infty$. In other papers [*3*] and [*4*] we derived expressions for the particular solutions of (1)

$$\begin{array}{lllll} P_k^{-m,n}(z), & P_k^{m,-n}(z), & P_k^{-m,-n}(z), & P^{m,n}_{k-1}(z), & Q_k^{-m,n}(z), \\ Q_k^{m,-n}(z), & Q_k^{-m,-n}(z), & Q^{m,n}_{-k-1}(z), & P_k^{n,m}(-z), & Q_k^{n,m}(-z) \end{array}$$

in terms of $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$.

By means of the well known linear transformation formulas of the hypergeometric functions the $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ are expressible in several ways in the forms

$$P_k^{m,n}(z) = A_1 F\{a_1, b_1; c_1; \zeta\} + A_2 F\{a_2, b_2; c_2; \zeta\} \quad |\zeta| < 1$$

and

$$Q_k^{m,n}(z) = A_3 F\{a_3, b_3; c_3; \zeta\} + A_4 F\{a_4, b_4; c_4; \zeta\} \quad |\zeta| < 1,$$

where ζ is a linear function of z which depends on the choice of the transformation. In [*5*] we carried out these expansions.

If the a, b and c in $F\{a, b; c; z\}$ are unrestricted, there exists no quadratic transformation of the hypergeometric function*. If and only if $\pm(1-c)$, $\pm(a-b)$, $\pm(a+b-c)$ have the property that one of them equals $\frac{1}{2}$ or that two of them are equal, there exists such a quadratic transformation, this being given by the formulas of GAUSS and KUMMER. For unrestricted values of k, m and n in $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ the conditions for quadratic transformation are not fulfilled, and therefore it is not possible to derive quadratic expansions of these functions in terms of single hypergeometric functions. We have deduced expressions for $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ in terms of series of generalized hypergeometric functions or MacRobert's E-functions. These results will be published elsewhere.

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* *I.e.*, a transformation of $F\{a, b; c; z\}$ in functions $F\{a', b'; c'; \eta\}$, in which the variables z and η are connected by a quadratic relation.

However, when the parameters m and n satisfy special conditions, quadratic expansions can be derived for the $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$. In the present paper I shall deduce expressions for these functions as finite sums of hypergeometric functions with quadratic forms of z as variable under the rather strong condition: *$n-m$ is an even integer or* 0.

In § 2 these expressions for $P_k^{m,n}(z)$ are deduced. In § 3 I shall prove a relation between $Q_k^{m,n}(z)$, $P_k^{-n,-m}(-z)$ and $P_k^{m,n}(z)$, by means of which the expansions for $Q_k^{m,n}(z)$ are deduced in § 4.

For $m=n$ these expressions lead to the well known quadratic expansions for the associated Legendre functions (see [*6*], Chapter III).

2. Expansions for $P_k^{m,n}(z)$

The starting-point is the relation [*1, 11*]:

$$P_k^{m,n}(z)=\frac{1}{\Gamma(1-m)}\,\frac{(z+1)^{\frac{n}{2}}}{(z-1)^{\frac{m}{2}}}\,F\left\{k+\frac{n-m}{2}+1,\ -k+\frac{n-m}{2};\ 1-m;\frac{1-z}{2}\right\},$$

valid for $|z-1|<2$. On account of the relation

$$\frac{d^n}{dz^n}F\{a,b;c;z\}=\frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c+n)}F\{a+n,b+n;c+n;z\},\tag{2}$$

when $n-m$ is an even integer or 0 we have

$$\begin{aligned}P_k^{m,n}(z)=\frac{(z+1)^{\frac{n}{2}}}{(z-1)^{\frac{m}{2}}}\,&\frac{\Gamma\left(k-\frac{n-m}{2}+1\right)2^{\frac{n-m}{2}}}{\Gamma\left(k+\frac{n-m}{2}+1\right)\Gamma\left(1-\frac{m+n}{2}\right)}\times\\&\times\frac{d^{\frac{n-m}{2}}}{dz^{\frac{n-m}{2}}}F\left\{k+1,-k;1-\frac{m+n}{2};\frac{1-z}{2}\right\}.\end{aligned}\tag{3}$$

Since

$$P_k^{m}(z)=\frac{1}{\Gamma(1-m)}\left(\frac{z+1}{z-1}\right)^{\frac{m}{2}}F\left\{k+1,-k;1-m;\frac{1-z}{2}\right\},\tag{4}$$

where $P_k^m(z)$ denotes the associate Legendre function of the first kind, the expression (3) may be written:

$$\begin{aligned}P_k^{m,n}(z)=\frac{(z+1)^{\frac{n}{2}}}{(z-1)^{\frac{m}{2}}}\,&\frac{\Gamma\left(k-\frac{n-m}{2}+1\right)2^{\frac{n-m}{2}}}{\Gamma\left(k+\frac{n-m}{2}+1\right)}\times\\&\times\frac{d^{\frac{n-m}{2}}}{dz^{\frac{n-m}{2}}}\left\{\left(\frac{z-1}{z+1}\right)^{\frac{m+n}{4}}P_k^{\frac{m+n}{2}}(z)\right\}.\end{aligned}\tag{5}$$

2.1. For $|z| < 1$ we have

$$P_k^m(z) = \frac{2^m \pi^{\frac{1}{2}} (z^2-1)^{-\frac{m}{2}}}{\Gamma(-\frac{1}{2}k - \frac{1}{2}m + \frac{1}{2})\, \Gamma(\frac{1}{2}k - \frac{1}{2}m + 1)} \times$$

$$\times F\left\{-\frac{1}{2}k - \frac{1}{2}m, \frac{1}{2}k - \frac{1}{2}m + \frac{1}{2}; \frac{1}{2}; z^2;\right\}$$

$$- \frac{2^{m+1} \pi^{\frac{1}{2}} z (z^2-1)^{-\frac{m}{2}}}{\Gamma(\frac{1}{2}k - \frac{1}{2}m + \frac{1}{2})\, \Gamma(-\frac{1}{2}k - \frac{1}{2}m)} \times$$

$$\times F\left\{-\frac{1}{2}k - \frac{1}{2}m + \frac{1}{2}, \frac{1}{2}k - \frac{1}{2}m + 1; \frac{3}{2}; z^2\right\}.$$

Hence (5) can be transformed into

$$\begin{aligned} P_k^{m,n}(z) = {} & \frac{(z+1)^{\frac{n}{2}}}{(z-1)^{\frac{m}{2}}} \frac{\Gamma\left(k - \frac{n-m}{2} + 1\right) 2^n \pi^{\frac{1}{2}}}{\Gamma\left(k + \frac{n-m}{2} + 1\right) \Gamma\left(-\frac{1}{2}k - \frac{m+n}{4} + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}k - \frac{m+n}{4} + 1\right)} \times \\ & \times \frac{d^{\frac{n-m}{2}}}{dz^{\frac{n-m}{2}}} \left[(z+1)^{-\frac{m+n}{2}} F\left\{-\frac{1}{2}k - \frac{m+n}{4}, \frac{1}{2}k - \frac{m+n}{4} + \frac{1}{2}; \frac{1}{2}; z^2\right\}\right] - \\ & - \frac{(z+1)^{\frac{n}{2}}}{(z-1)^{\frac{m}{2}}} \frac{\Gamma\left(k - \frac{n-m}{2} + 1\right) 2^{n+1} \pi^{\frac{1}{2}}}{\Gamma\left(k + \frac{n-m}{2} + 1\right) \Gamma\left(\frac{1}{2}k - \frac{m+n}{4} + \frac{1}{2}\right) \Gamma\left(-\frac{1}{2}k - \frac{m+n}{4}\right)} \times \\ & \times \frac{d^{\frac{n-m}{2}}}{dz^{\frac{n-m}{2}}} \left[z(z+1)^{-\frac{m+n}{2}} F\left\{-\frac{1}{2}k - \frac{m+n}{4} + \frac{1}{2}, \frac{1}{2}k - \frac{m+n}{4} + 1; \frac{3}{2}; z^2\right\}\right]. \end{aligned} \tag{6}$$

We find, on carrying out the differentiation and employing the relations

$$\Gamma(z)\, \Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

and

$$\Gamma(2z) = 2^{2z-1} \pi^{-\frac{1}{2}} \Gamma(z)\, \Gamma(z + \tfrac{1}{2}),$$

that

$$P_k^{m,n}(z) = A_1(m, n, k, z) + A_2(m, n, k, z), \tag{7}$$

where

$$\begin{aligned} & A_1(m, n, k, z) \\ & = \frac{(-1)^{\frac{n-m}{2}} 2^{-m-1} \Gamma\left(\frac{n-m}{2} + 1\right) \Gamma\left(k - \frac{n-m}{2} + 1\right) (z+1)^{-\frac{n}{2}} (z-1)^{-\frac{m}{2}}}{\Gamma\left(\frac{m+n}{2}\right) \Gamma\left(k - \frac{m+n}{2} + 1\right) \Gamma\left(k + \frac{n-m}{2} + 1\right) \Gamma\left(-k - \frac{m+n}{2}\right)} \times \\ & \times \sum_{r=0}^{\frac{n-m}{2}} \frac{\Gamma(n-r)(z+1)^r (-1)^r}{\Gamma\left(\frac{n-m}{2} - r + 1\right)} \times \\ & \times \sum_{q=0}^{\left[\frac{r}{2}\right]} \frac{(2z)^{r-2q} \Gamma\left(-\frac{1}{2}k - \frac{m+n}{4} + r - q\right) \Gamma\left(\frac{1}{2}k - \frac{m+n}{4} + r - q + \frac{1}{2}\right)}{\Gamma(r - 2q + 1)\, \Gamma(q+1)\, \Gamma\left(\frac{1}{2} + r - q\right)} \times \\ & \times F\left\{-\frac{1}{2}k - \frac{m+n}{4} + r - q, \frac{1}{2}k - \frac{m+n}{4} + r - q + \frac{1}{2}; \frac{1}{2} + r - q; z^2\right\}, \end{aligned} \tag{8}$$

and

$$A_2(m, n, k, z)$$

$$= \frac{(-1)^{\frac{n-m}{2}+1}\, 2^{-m-1}\, \Gamma\left(\frac{n-m}{2}+1\right) \Gamma\left(k-\frac{n-m}{2}+1\right) (z+1)^{-\frac{n}{2}} (z-1)^{-\frac{m}{2}}}{\Gamma\left(\frac{m+n}{2}\right) \Gamma\left(k-\frac{m+n}{2}+1\right) \Gamma\left(k+\frac{n-m}{2}+1\right) \Gamma\left(-k-\frac{m+n}{2}\right)} \times$$

$$(9) \qquad \times \sum_{r=0}^{\frac{n-m}{2}} \frac{\Gamma(n-r-1)\,(z+1)^r\,(-1)^r \left\{z\left(\frac{n+m}{2}-1\right)-\left(\frac{n-m}{2}-r\right)\right\}}{\Gamma\left(\frac{n-m}{2}-r+1\right)} \times$$

$$\times \sum_{q=0}^{\left[\frac{r}{2}\right]} \frac{(2z)^{r-2q}\, \Gamma\left(-\frac{1}{2}k-\frac{m+n}{4}+r-q+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}k-\frac{m+n}{4}+r-q+1\right)}{\Gamma(r-2q+1)\,\Gamma(q+1)\,\Gamma\left(\frac{3}{2}+r-q\right)} \times$$

$$\times F\left\{-\frac{1}{2}k-\frac{m+n}{4}+r-q+\frac{1}{2}, \frac{1}{2}k-\frac{m+n}{4}+r-q+1; \frac{3}{2}+r-q;\ z^2\right\};$$

these results hold when $|z| < 1$.

2.2. If we substitute in (5) for $P_k^{\frac{m+n}{2}}(z)$ the expression

$$(10) \qquad P_k^{\frac{m+n}{2}}(z) = \frac{2^{\frac{m+n}{2}} (z^2-1)^{-\frac{m+n}{4}}}{\Gamma\left(1-\frac{m+n}{2}\right)} \times$$

$$\times F\left\{\frac{1}{2}k-\frac{m+n}{4}+\frac{1}{2},\ -\frac{1}{2}k-\frac{m+n}{4};\ 1-\frac{m+n}{2};\ 1-z^2\right\},$$

valid for $|1-z^2| < 1$, $\Re z > 0$, and follow the same argument, we obtain:

$$P_k^{m,n}(z) = B(m, n, k, z)$$

$$= \frac{(-1)^{\frac{n-m}{2}}\, \Gamma\left(\frac{n-m}{2}+1\right) \Gamma\left(k-\frac{n-m}{2}+1\right) 2^n (z-1)^{-\frac{m}{2}} (z+1)^{-\frac{n}{2}}}{\Gamma\left(\frac{m+n}{2}\right) \Gamma\left(k+\frac{n-m}{2}+1\right) \Gamma\left(\frac{1}{2}k-\frac{m+n}{4}+\frac{1}{2}\right) \Gamma\left(-\frac{1}{2}k-\frac{m+n}{4}\right)} \times$$

$$\times \sum_{r=0}^{\frac{n-m}{2}} \frac{\Gamma(n-r)\,(z+1)^r}{\Gamma\left(\frac{n-m}{2}-r+1\right)} \times$$

$$(11) \qquad \times \sum_{q=0}^{\left[\frac{r}{2}\right]} \frac{(-1)^q\,(2z)^{r-2q}\, \Gamma\left(\frac{1}{2}k-\frac{m+n}{4}+r-q+\frac{1}{2}\right) \Gamma\left(-\frac{1}{2}k-\frac{m+n}{4}+r-q\right)}{\Gamma(q+1)\,\Gamma(r-2q+1)\,\Gamma\left(1-\frac{m+n}{2}+r-q\right)} \times$$

$$\times F\left\{\frac{1}{2}k-\frac{m+n}{4}+r-q+\frac{1}{2}, -\frac{1}{2}k-\frac{m+n}{4}+r-q;\ 1-\frac{m+n}{2}+r-q; 1-z^2\right\},$$

which is valid in the same z-region.

2.3. Applying to the hypergeometric functions in (11) the quadratic transformation

$$(12) \qquad F\{a, b; c; z\} = \frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)} (-z)^{-a} F\left\{a, 1-c+a; 1-b+a; \frac{1}{z}\right\}$$

$$+ \frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)} (-z)^{-b} F\left\{b, 1-c+b;\ 1-a+b; \frac{1}{2}\right\}, \quad |\arg(-z)| < \pi,$$

we find after some calculation that

$$P_k^{m,n}(z) = C(m, n, k, z) + C(m, n, -k-1, z), \tag{13}$$

where

$C\,(m, n, k, z)$

$$= \frac{(-1)^{\frac{n-m}{2}} \pi^{-\frac{1}{2}} 2^{-k+\frac{n-m}{2}-1} \Gamma\left(\frac{n-m}{2}+1\right) \Gamma\left(k-\frac{n-m}{2}+1\right) \Gamma\left(-k-\frac{1}{2}\right) \times}{\Gamma\left(\frac{m+n}{2}\right) \Gamma\left(k+\frac{n-m}{2}+1\right) \Gamma\left(\frac{1}{2}k-\frac{m+n}{4}+\frac{1}{2}\right) \times}$$

$$\frac{\times\,(z-1)^{-\frac{1}{2}k+\frac{n-m}{4}-\frac{1}{2}} (z+1)^{-\frac{1}{2}k-\frac{n-m}{4}-\frac{1}{2}}}{\times\,\Gamma\left(-k-\frac{m+n}{2}\right)} \sum_{r=0}^{\frac{n-m}{2}} \frac{\Gamma(n-r)}{\Gamma\left(\frac{n-m}{2}-r+1\right)} \times \tag{14}$$

$$\times \sum_{q=0}^{\left[\frac{r}{2}\right]} \frac{(-1)^q (2z)^{r-2q} (z+1)^q (z-1)^{-r+q}\, \Gamma\left(\frac{1}{2}k-\frac{m+n}{4}+r-q+\frac{1}{2}\right)}{\Gamma\,(q+1)\, \Gamma(r-2q+1)} \times$$

$$\times F\left\{\frac{1}{2}k-\frac{m+n}{4}+r-q+\frac{1}{2},\ \frac{1}{2}k+\frac{m+n}{4}+\frac{1}{2}; k+\frac{3}{2};\ \frac{1}{1-z^2}\right\};$$

by analytic continuation, this formula holds when $|1-z^2|>1$.

2.4. If we transform the hypergeometric functions in (11) by means of

$$F\{a, b; c; z\} = (1-z)^{-b} F\left\{b, c-a; c; \frac{z}{z-1}\right\}, \tag{15}$$

we obtain

$P_k^{m,n}(z) = D\,(m, n, k, z)$

$$= \frac{(-1)^{\frac{n-m}{2}} 2^n \Gamma\left(\frac{n-m}{2}+1\right) \Gamma\left(k-\frac{n-m}{2}+1\right) (z-1)^{-\frac{m}{2}} (z+1)^{-\frac{n}{2}} z^{k+\frac{m+n}{2}}}{\Gamma\left(\frac{m+n}{2}\right) \Gamma\left(k+\frac{n-m}{2}+1\right) \Gamma\left(\frac{1}{2}k-\frac{m+n}{4}+\frac{1}{2}\right) \Gamma\left(-\frac{1}{2}k-\frac{m+n}{4}\right)} \times$$

$$\times \sum_{r=0}^{\frac{n-m}{2}} \frac{\Gamma(n-r)\,(z+1)^r z^{-r}}{\Gamma\left(\frac{n-m}{2}-r+1\right)} \times \tag{16}$$

$$\times \sum_{q=0}^{\left[\frac{r}{2}\right]} \frac{(-1)^q 2^{r-2q} \Gamma\left(\frac{1}{2}k-\frac{m+n}{4}+r-q+\frac{1}{2}\right) \Gamma\left(-\frac{1}{2}k-\frac{m+n}{4}+r-q\right)}{\Gamma(q+1)\, \Gamma(r-2q+1)\, \Gamma\left(1-\frac{m+n}{2}+r-q\right)} \times$$

$$\times F\left\{-\frac{1}{2}k-\frac{m+n}{4}+\frac{1}{2}, -\frac{1}{2}k-\frac{m+n}{4}+r-q; 1-\frac{m+n}{2}+r-q; 1-\frac{1}{z^2}\right\},$$

which holds when $\left|1-\frac{1}{z^2}\right|<1$, $\Re z>0$.

2.5. On applying (15) to the hypergeometric functions in (13), we have

$$P_k^{m,n}(z) = G(m, n, k, z) + G(m, n, -k-1, z), \tag{17}$$

where

$$
\begin{aligned}
G(m, n, k, z) &= \frac{(-1)^{\frac{n-m}{2}} \pi^{-\frac{1}{2}} 2^{-k+\frac{n-m}{2}-1} \Gamma\left(\frac{n-m}{2}+1\right) \Gamma\left(k-\frac{n-m}{2}+1\right) \times}{\Gamma\left(\frac{m+n}{2}\right) \Gamma\left(k+\frac{n-m}{2}+1\right) \Gamma\left(\frac{1}{2} k-\frac{m+n}{4}+\frac{1}{2}\right) \times} \\
&\quad \frac{\times \Gamma\left(-k-\frac{1}{2}\right) (z-1)^{\frac{n}{2}} (z+1)^{\frac{m}{2}} z^{-k-\frac{m+n}{2}-1}}{\times \Gamma\left(-k-\frac{m+n}{2}\right)} \sum_{r=0}^{\frac{n-m}{2}} \frac{\Gamma(n-r)}{\Gamma\left(\frac{n-m}{2}-r+1\right)} \times \\
&\quad \times \sum_{q=0}^{\left[\frac{r}{2}\right]} \frac{(-1)^q (2z)^{r-2q} (z+1)^q (z-1)^{-r+q} \Gamma\left(\frac{1}{2} k-\frac{m+n}{4}+r-q+\frac{1}{2}\right)}{\Gamma(q+1) \Gamma(r-2q+1)} \times \\
&\quad \times F\left\{\frac{1}{2} k+\frac{m+n}{4}+\frac{1}{2},\ \frac{1}{2} k+\frac{m+n}{4}-r+q+1 ; k+\frac{3}{2} ;\ \frac{1}{z^2}\right\},
\end{aligned} \tag{18}
$$

which holds when $|z|>1$.

2.6. Furthermore on applying (12) also to the hypergeometric functions in (16) we obtain

$$P_k^{m, n}(z)=H_1(m, n, k, z)+H_2(m, n, k, z), \tag{19}$$

where

$$
\begin{aligned}
&H_1(m, n, k, z) \\
&= \frac{(-1)^{\frac{n-m}{2}} e^{\mp \pi i\left(\frac{1}{2} k+\frac{m+n}{4}-\frac{1}{2}\right)} 2^n \Gamma\left(\frac{n-m}{2}+1\right) \Gamma\left(k-\frac{n-m}{2}+1\right)}{\Gamma\left(\frac{m+n}{2}\right) \Gamma\left(k+\frac{n-m}{2}+1\right) \Gamma\left(\frac{1}{2} k-\frac{m+n}{4}+\frac{1}{2}\right) \Gamma\left(-\frac{1}{2} k-\frac{m+n}{4}\right)} \times \\
&\quad \times (z-1)^{\frac{1}{2} k+\frac{n-m}{4}-\frac{1}{2}} (z+1)^{\frac{1}{2} k-\frac{n-m}{4}-\frac{1}{2}} \times \\
&\quad \times \sum_{r=0}^{\frac{n-m}{2}} \frac{\Gamma(n-r) (z+1)^r z^{1-r}}{\Gamma\left(\frac{n-m}{2}-r+1\right)} \sum_{q=0}^{\left[\frac{r}{2}\right]} \frac{(-1)^q 2^{r-2q} \Gamma\left(r-q-\frac{1}{2}\right)}{\Gamma(q+1) \Gamma(r-2q+1)} \times \\
&\quad \times F\left\{-\frac{1}{2} k-\frac{m+n}{4}+\frac{1}{2}, -\frac{1}{2} k+\frac{m+n}{4}+\frac{1}{2}-r+q ; \frac{3}{2}-r+q ; \frac{z^2}{z^2-1}\right\}
\end{aligned} \tag{20}
$$

and

$$
\begin{aligned}
H_2(m, n, k, z) &= \frac{(-1)^{\frac{n-m}{2}} e^{\mp \pi i\left(\frac{1}{2} k+\frac{m+n}{4}\right)} 2^{-m-1} \Gamma\left(\frac{n-m}{2}+1\right) \Gamma\left(k-\frac{n-m}{2}+1\right)}{\Gamma\left(\frac{m+n}{2}\right) \Gamma\left(k+\frac{n-m}{2}+1\right) \Gamma\left(k-\frac{m+n}{2}+1\right) \Gamma\left(-k-\frac{m+n}{2}\right)} \times \\
&\quad \times (z-1)^{\frac{1}{2} k+\frac{n-m}{4}} (z+1)^{\frac{1}{2} k-\frac{n-m}{4}} \times \\
&\quad \times \sum_{r=0}^{\frac{n-m}{2}} \frac{\Gamma(n-r)}{\Gamma\left(\frac{n-m}{2}-r+1\right)} \sum_{q=0}^{\left[\frac{r}{2}\right]} \frac{(-1)^q (2z)^{r-2q} (z+1)^q (z-1)^{-r+q} \times}{\Gamma(q+1) \times} \\
&\quad \frac{\times \Gamma\left(\frac{1}{2} k-\frac{m+n}{4}+r-q+\frac{1}{2}\right) \Gamma\left(-\frac{1}{2} k-\frac{m+n}{4}+r-q\right)}{\times \Gamma(r-2q+1) \Gamma\left(\frac{1}{2}+r-q\right)} \times \\
&\quad \times F\left\{-\frac{1}{2} k-\frac{m+n}{4}+r-q, -\frac{1}{2} k+\frac{m+n}{4} ;\ \frac{1}{2}+r-q ;\ \frac{z^2}{z^2-1}\right\} ;
\end{aligned} \tag{21}
$$

the upper or lower sign of the exponential being taken according as $\Im z > 0$ or < 0. Formula (19) is valid when $\left|\dfrac{z^2}{z^2-1}\right| < 1$.

3. Relation between $P_k^{-n,-m}(-z)$, $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$

Theorem. *When $n-m$ is an even integer or 0, we have the relation*

$$
\begin{aligned}
e^{-\pi i m}\, Q_k^{m,n}(z) = & -\frac{\pi e^{\pm \pi i m}}{2 \sin \pi k} \times \\
& \times \left\{ \frac{\Gamma\left(k+\frac{m+n}{2}+1\right)}{\Gamma\left(k-\frac{m+n}{2}+1\right)}\, P_k^{-n,-m}(-z) - e^{\mp \pi i k}\, P_k^{m,n}(z) \right\}.
\end{aligned}
\tag{22}
$$

Proof. Using the relations

$$
\begin{aligned}
P_k^{-m,-n}(z) = & \frac{1}{2^{n-m}} \frac{\Gamma\left(k-\frac{m+n}{2}+1\right)\Gamma\left(k+\frac{n-m}{2}+1\right)}{\Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma\left(k-\frac{n-m}{2}+1\right)} \times \\
& \times \left\{ P_k^{m,n}(z) - \frac{2}{\pi} \sin m\pi\, e^{-\pi i m}\, Q_k^{m,n}(z) \right\} \qquad [4, (18)],
\end{aligned}
$$

$$
\begin{aligned}
P_k^{n,m}(-z) = & \frac{e^{\pm \frac{n-m}{2}\pi i}\, \Gamma\left(k+\frac{n-m}{2}+1\right)}{2^{n-m}\, \Gamma\left(k-\frac{n-m}{2}+1\right)} \times \\
& \times \left\{ -\frac{2}{\pi} \sin\left(k+\frac{m+n}{2}\right)\pi\, e^{-\pi i m}\, Q_k^{m,n}(z) + e^{\mp \pi i \left(k+\frac{n-m}{2}\right)}\, P_k^{m,n}(z) \right\} \qquad [4, (21)]
\end{aligned}
$$

and

$$
Q_k^{n,m}(-z) = -\frac{e^{\pi i(\pm k - m + n)}}{2^{n-m}} \frac{\Gamma\left(k+\frac{n-m}{2}+1\right)}{\Gamma\left(k-\frac{n-m}{2}+1\right)}\, Q_k^{m,n}(z) \qquad [4, (25)],
$$

we find

$$
\begin{aligned}
P_k^{-n,-m}(-z) = & \frac{\Gamma\left(k-\frac{m+n}{2}+1\right)}{\Gamma\left(k+\frac{m+n}{2}+1\right)} \left[\frac{2}{\pi} e^{-\pi i m}\, Q_k^{m,n}(z) \left\{ -\sin\left(k+\frac{m+n}{2}\right)\pi \times \right.\right. \\
& \left.\left. \times e^{\pm \frac{n-m}{2}\pi i} + e^{\pm \pi i k} \sin n\pi \right\} + e^{\mp \pi i k}\, P_k^{m,n}(z) \right] \\
= & \frac{\Gamma\left(k-\frac{m+n}{2}+1\right)}{\Gamma\left(k+\frac{m+n}{2}+1\right)} \times \\
& \times \left\{ -\frac{2}{\pi} e^{-\pi i m}\, Q_k^{m,n}(z) \sin \pi k\, e^{\mp \pi i m} + e^{\mp \pi i k}\, P_k^{m,n}(z) \right\}.
\end{aligned}
$$

By solving for $Q_k^{m,n}(z)$ we have (22).

4. Expansions for $Q_k^{m,n}(z)$

It has been shown in [5, (8)] that

$$\begin{aligned}
&e^{-\pi i m} Q_k^{m,n}(z) \\
&\quad = \frac{2^{n-m-1}\,\Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma\left(k-\frac{n-m}{2}+1\right)\Gamma(-m)}{\Gamma\left(k+\frac{n-m}{2}+1\right)\Gamma\left(k-\frac{m+n}{2}+1\right)}\,(z+1)^{-\frac{n}{2}}(z-1)^{\frac{m}{2}}\times \\
&\qquad \times F\left\{-k-\frac{n-m}{2},\,k-\frac{n-m}{2}+1;\;1+m;\;\frac{1-z}{2}\right\}+ \\
&\qquad + \frac{1}{2}\,\Gamma(m)\,(z+1)^{\frac{n}{2}}(z-1)^{-\frac{m}{2}}\times \\
&\qquad \times F\left\{-k+\frac{n-m}{2},\,k+\frac{n-m}{2}+1;\;1-m;\;\frac{1-z}{2}\right\}.
\end{aligned}\tag{23}$$

Employing the well known relation

$$\begin{aligned}
&\frac{d^n}{dz^n}\left\{z^{c-1}(1-z)^{a+b-c}F\{a,b;c;z\}\right\} \\
&\qquad\qquad = \frac{\Gamma(c)}{\Gamma(c-n)}\,z^{c-1-n}(1-z)^{a+b-c-n}F\{a-n,\,b-n;\,c-n;\,z\}
\end{aligned}$$

and (4), we find that

$$\begin{aligned}
&F\left\{-k-\frac{n-m}{2},\,k-\frac{n-m}{2}+1;\,1+m;\;\frac{1-z}{2}\right\} \\
&\qquad = 2^{-\frac{n-m}{2}}\,\Gamma(1+m)\,\frac{(z+1)^n}{(z-1)^m}\,\frac{d^{\frac{n-m}{2}}}{dz^{\frac{n-m}{2}}}\left\{\left(\frac{z-1}{z+1}\right)^{\frac{m+n}{4}}P_k^{-\frac{m+n}{2}}(z)\right\},
\end{aligned}\tag{24}$$

and, on account of (2) and (4),

$$\begin{aligned}
&F\left\{-k+\frac{n-m}{2},\,k+\frac{n-m}{2}+1;\;1-m;\;\frac{1-z}{2}\right\} \\
&\qquad = \frac{2^{\frac{n-m}{2}}\,\Gamma(1-m)\,\Gamma\left(k-\frac{n-m}{2}+1\right)}{\Gamma\left(k+\frac{n-m}{2}+1\right)}\,\frac{d^{\frac{n-m}{2}}}{dz^{\frac{n-m}{2}}}\left\{\left(\frac{z-1}{z+1}\right)^{\frac{m+n}{4}}P_k^{\frac{m+n}{2}}(z)\right\}.
\end{aligned}\tag{25}$$

From (23), (24) and (25) we obtain

$$\begin{aligned}
&e^{-\pi i m} Q_k^{m,n}(z) = \frac{\pi\, 2^{\frac{n-m}{2}-1}\,\Gamma\left(k-\frac{n-m}{2}+1\right)}{\Gamma\left(k+\frac{n-m}{2}+1\right)\sin m\pi}\,\frac{(z+1)^{\frac{n}{2}}}{(z-1)^{\frac{m}{2}}}\times \\
&\qquad \times\left[-\frac{\Gamma\left(k+\frac{m+n}{2}+1\right)}{\Gamma\left(k-\frac{m+n}{2}+1\right)}\,\frac{d^{\frac{n-m}{2}}}{dz^{\frac{n-m}{2}}}\left\{\left(\frac{z-1}{z+1}\right)^{\frac{m+n}{4}}P_k^{-\frac{m+n}{2}}(z)\right\}+\right. \\
&\qquad \left. +\,\frac{d^{\frac{n-m}{2}}}{dz^{\frac{n-m}{2}}}\left\{\left(\frac{z-1}{z+1}\right)^{\frac{m+n}{4}}P_k^{\frac{m+n}{2}}(z)\right\}\right].
\end{aligned}\tag{26}$$

4.1. Since $-m \geqq -n$, we may apply (22) to (7), and we find that

$$e^{-\pi i m} Q_k^{m,n}(z) = -\frac{\pi e^{\pm \pi i m}}{2 \sin \pi k} \left[\frac{\Gamma\left(k + \frac{m+n}{2} + 1\right)}{\Gamma\left(k - \frac{m+n}{2} + 1\right)} \{A_1(-n, -m, k, -z) + \right.$$

$$\left. + A_2(-n, -m, k, -z)\} - e^{\mp \pi i k} \{A_1(m, n, k, z) + A_2(m, n, k, z)\} \right],$$

where $A_1(m, n, k, z)$ and $A_2(m, n, k, z)$ are defined in (8) and (9) respectively.

From (6), (8) and (9) it follows that

$$A_1(m, n, k, z)$$

$$= \frac{(z+1)^{\frac{n}{2}}}{(z-1)^{\frac{m}{2}}} \frac{\Gamma\left(k - \frac{n-m}{2} + 1\right) 2^n \pi^{\frac{1}{2}}}{\Gamma\left(k + \frac{n-m}{2} + 1\right) \Gamma\left(-\frac{1}{2} k - \frac{m+n}{4} + \frac{1}{2}\right) \Gamma\left(-\frac{1}{2} k - \frac{m+n}{4} + 1\right)} \times$$

$$\times \frac{d^{\frac{n-m}{2}}}{dz^{\frac{n-m}{2}}} \left[(z+1)^{-\frac{m+n}{2}} F\left\{-\frac{1}{2} k - \frac{m+n}{4}, \frac{1}{2} k - \frac{m+n}{4} + \frac{1}{2}; \frac{1}{2}; z^2\right\} \right].$$

$$A_2(m, n, k, z)$$

$$= -\frac{(z+1)^{\frac{n}{2}}}{(z-1)^{\frac{m}{2}}} \frac{\Gamma\left(k - \frac{n-m}{2} + 1\right) 2^{n+1} \pi^{\frac{1}{2}}}{\Gamma\left(k + \frac{n-m}{2} + 1\right) \Gamma\left(\frac{1}{2} k - \frac{m+n}{4} + \frac{1}{2}\right) \Gamma\left(-\frac{1}{2} k - \frac{m+n}{4}\right)} \times$$

$$\times \frac{d^{\frac{n-m}{2}}}{dz^{\frac{n-m}{2}}} \left[z(z+1)^{-\frac{m+n}{2}} F\left\{-\frac{1}{2} k - \frac{m+n}{4} + \frac{1}{2}, \frac{1}{2} k - \frac{m+n}{4} + 1; \frac{3}{2}; z^2\right\} \right].$$

By changing (m, n, z) into $(-n, -m, -z)$ we find after some reduction:

$$A_1(-n, -m, k, -z)$$

$$= \frac{\Gamma\left(-\frac{1}{2} k - \frac{m+n}{4} + \frac{1}{2}\right) \Gamma\left(\frac{1}{2} k - \frac{m+n}{4} + 1\right) 2^{-m-n}}{\Gamma\left(-\frac{1}{2} k + \frac{m+n}{4} + \frac{1}{2}\right) \Gamma\left(\frac{1}{2} k + \frac{m+n}{4} + 1\right)} A_1(m, n, k, z),$$

$$A_2(-n, -m, k, -z)$$

$$= -\frac{\Gamma\left(\frac{1}{2} k - \frac{m+n}{4} + \frac{1}{2}\right) \Gamma\left(-\frac{1}{2} k - \frac{m+n}{4}\right) 2^{-m-n}}{\Gamma\left(\frac{1}{2} k + \frac{m+n}{4} + \frac{1}{2}\right) \Gamma\left(-\frac{1}{2} k + \frac{m+n}{4}\right)} A_2(m, n, k, z),$$

and thus

$$\frac{\Gamma\left(k + \frac{m+n}{2} + 1\right)}{\Gamma\left(k - \frac{m+n}{2} + 1\right)} A_1(-n, -m, k, -z) - e^{\mp \pi i k} A_1(m, n, k, z)$$

$$= \left\{ \frac{\sin\left(\frac{1}{2} k - \frac{m+n}{4} + \frac{1}{2}\right)\pi}{\sin\left(\frac{1}{2} k + \frac{m+n}{4} + \frac{1}{2}\right)\pi} - e^{\mp \pi i k} \right\} A_1(m, n, k, z)$$

$$= -\frac{e^{\mp \pi i \left(\frac{m+n}{4} + \frac{1}{2} k + \frac{1}{2}\right)} \sin \pi k}{\cos\left(\frac{1}{2} k + \frac{m+n}{4}\right)\pi} A_1(m, n, k, z).$$

In a similar way we find

$$\frac{\Gamma\left(k+\frac{m+n}{2}+1\right)}{\Gamma\left(k-\frac{m+n}{2}+1\right)} A_2(-n,-m,k,-z) - e^{\mp\pi i k} A_2(m,n,k,z)$$
$$= -\frac{e^{\mp\pi i\left(\frac{m+n}{4}+\frac{1}{2}k\right)}}{\sin\left(\frac{1}{2}k+\frac{m+n}{4}\right)\pi} \sin\pi k\, A_2(m,n,k,z).$$

so that

$$(27)\qquad e^{-\pi i m} Q_k^{m,n}(z) = \frac{\pi}{2}\,\frac{e^{\mp\pi i\left(-\frac{3m-n}{4}+\frac{1}{2}k+\frac{1}{2}\right)}}{\cos\left(\frac{1}{2}k+\frac{m+n}{4}\right)\pi} A_1(m,n,k,z) + {}$$
$$+\frac{\pi}{2}\,\frac{e^{\mp\pi i\left(-\frac{3m-n}{4}+\frac{1}{2}k\right)}}{\sin\left(\frac{1}{2}k+\frac{m+n}{4}\right)\pi} A_2(m,n,k,z).$$

4.2. To derive an expansion of $Q_k^{m,n}(z)$ in hypergeometric functions in $(1-z^2)$ the method used in § 4.1. fails, the formula (10) not being valid for $\Re(z)<0$. Therefore in this case we shall use the more direct method.

From (26) and (10) we deduce

$$e^{-\pi i m} Q_k^{m,n}(z)$$
$$= \frac{\pi\,\Gamma\left(k-\frac{n-m}{2}+1\right)}{\Gamma\left(k+\frac{n-m}{2}+1\right)\sin m\pi}\,\frac{(z+1)^{\frac{n}{2}}}{(z-1)^{\frac{n}{2}}}\left[-\frac{2^{-m-1}\,\Gamma\left(k+\frac{m+n}{2}+1\right)}{\Gamma\left(k-\frac{m+n}{2}+1\right)\Gamma\left(1+\frac{m+n}{2}\right)}\times\right.$$
$$\times\frac{d^{\frac{n-m}{2}}}{dz^{\frac{n-m}{2}}}\left\{(z-1)^{\frac{m+n}{2}} F\left\{\frac{1}{2}k+\frac{m+n}{4}+\frac{1}{2},\right.\right.$$
$$\left.\left.-\frac{1}{2}k+\frac{m+n}{4};\ \frac{m+n}{2}+1;\ 1-z^2\right\}\right\}+$$
$$+\frac{2^{n-1}}{\Gamma\left(1-\frac{m+n}{2}\right)}\,\frac{d^{\frac{n-m}{2}}}{dz^{\frac{n-m}{2}}}\left\{(z+1)^{-\frac{m+n}{2}} F\left\{\frac{1}{2}k-\frac{m+n}{4}+\frac{1}{2},\right.\right.$$
$$\left.\left.\left.-\frac{1}{2}k-\frac{m+n}{4};\ 1-\frac{m+n}{2};\ 1-z^2\right\}\right\}\right].$$

On carrying out the differentiations we find that

$$(28)\qquad e^{-\pi i m} Q_k^{m,n}(z) = \frac{\pi}{2\sin m\pi}\times$$
$$\times\left\{B(m,n,k,z) - \frac{\Gamma\left(k+\frac{m+n}{2}+1\right)e^{\pm\pi i\frac{m+n}{2}}}{\Gamma\left(k-\frac{m+n}{2}+1\right)} B(-n,-m,k,-z)\right\},$$

where $B(m, n, k, z)$ is defined in (11). Formula (28) holds good when $|1-z^2|<1$ and $\Re z>0$.

4.3. On substituting (13) in (22) and following the argument used in § 4.1., we find that

$$e^{-\pi i m}\, Q_k^{m,n}(z) = \frac{(-1)^{\frac{n-m}{2}}\, \pi \cos k\pi}{\sin\left(k+\frac{m+n}{2}\right)\pi}\, C(m, n, k, z), \tag{29}$$

where $C(m, n, k, z)$ is defined in (14). Formula (29) is valid when $|1-z^2|>1$.

4.4. On applying the transformation (15) to the hypergeometric functions in (28), we find that

$$\begin{aligned} e^{-\pi i m}\, Q_k^{m,n}(z) &= \frac{\pi}{2\sin m\pi}\times \\ &\times\left\{D(m, n, k, z) + \frac{\Gamma\left(k+\frac{m+n}{2}+1\right)e^{\mp\pi i k}}{\Gamma\left(k-\frac{m+n}{2}+1\right)}\, D(-n, -m, -k-1, -z)\right\}, \end{aligned} \tag{30}$$

where $D(m, n, k, z)$ is defined in (16), and where $\left|1-\frac{1}{z^2}\right|<1$.

4.5. Substituting (17) in (22) and following the argument of § 4.1., we obtain

$$e^{-\pi i m}\, Q_k^{m,n}(z) = \frac{(-1)^{\frac{n-m}{2}}\, \pi \cos k\pi}{\sin\left(k+\frac{m+n}{2}\right)\pi}\, G(m, n, k, z), \tag{31}$$

where $G(m, n, k, z)$ is defined in (18), and where $|z|>1$.

4.6. In a similar manner we find, substituting (19) in (22), after some calculation

$$\begin{aligned} e^{-\pi i m}\, Q_k^{m,n}(z) &= \frac{\pi}{2}\, \frac{e^{\pm\pi i\left(\frac{3m-n}{4}+\frac{1}{2}k\right)}}{\sin\left(\frac{1}{2}k+\frac{m+n}{4}\right)\pi}\, H_1(m, n, k, z) \\ &+ \frac{\pi}{2}\, \frac{e^{\mp\pi i\left(-\frac{3m-n}{4}+\frac{1}{2}k+\frac{1}{2}\right)}}{\cos\left(\frac{1}{2}k+\frac{m+n}{4}\right)\pi}\, H_2(m, n, k, z), \end{aligned} \tag{32}$$

where $H_1(m, n, k, z)$ and $H_2(m, n, k, z)$ are defined in (20) and (21), respectively, and where $\left|\frac{z^2}{z^2-1}\right|<1$.

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On the Convergence of the Rayleigh Quotient Iteration for the Computation of the Characteristic Roots and Vectors. V

(Usual Rayleigh Quotient for Non-Hermitian Matrices and Linear Elementary Divisors)

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58. In the case of a non-symmetric matrix, use of the generalized Rayleigh quotient appears to be particularly conformable to the problem, and in any case much more so than the use of the Rayleigh quotient in its original form. However, in some computations carried out in the mathematical laboratory of the Ramo-Wooldridge Corporation a routine implying the usual Rayleigh quotient has been used by Dr. Frank and turned out to converge quite well. It appeared of interest therefore to discuss the application of the usual Rayleigh quotient to the determination of the eigenvalues of an arbitrary (not necessarily Hermitian) matrix, and that will be done in this part of our series of papers. The main result is that the convergence is indeed essentially quadratic, if to the eigenvalue in question correspond only linear elementary divisors. Thus the result appears to be weaker than that achieved in the case of the generalized Rayleigh quotient iteration, where we get a cubic convergence. However, if we denote in this connection the computation implied in the solution of one linear algebraic system of order n as a "Horner", the application of the generalized Rayleigh quotient implies spending two Horners, while in using the ordinary Rayleigh quotient iteration we can achieve with two Horners two consecutive steps and therefore get the fourth degree of the error. From this point of view, even in the case of a non-Hermitian matrix, the use of the ordinary Rayleigh quotient iteration appears to be not only permissible but even advisable.

59. For an $n \times n$ matrix A, we consider the iteration procedure starting with a column-vector ξ_0 and an approximation λ_0 for the eigenvalue and continuing according to the rule

$$\xi_\nu = (A - \lambda_{\nu-1} I)^{-1} \xi_{\nu-1}, \tag{212}$$

$$\lambda_\nu = \frac{\xi_\nu^* A \xi_\nu}{\xi_\nu^* \xi_\nu}. \qquad (\nu = 1, 2, \ldots) \tag{213}$$

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Let σ be an eigenvalue of A of multiplicity m, to which correspond only linear elementary divisors. Then we shall prove

Theorem 1. *There exists a positive* $\delta^{(0)} = \delta(A)$, *depending only on* A, *and a matrix* H, *depending only on* A *and* σ, *such that if we have*

$$|\lambda_0 - \sigma| \leqq \delta^{(0)}, \tag{214}$$

and if for an eigenvector η *of* A *from the right belonging to* σ *the relation*

$$|\xi_0 - \eta| \leqq \delta^{(0)} |\eta| \tag{215}$$

holds, then the sequence λ_ν *tends to* σ.

Theorem 2. *If the sequence* λ_ν *tends to* σ *without any of the* λ_ν *becoming equal to one of other eigenvalues of* A, *and if* ξ_0 *satisfies* (215), *then we have*

$$\lambda_{\nu+1} - \sigma = o(\lambda_\nu - \sigma) \qquad (\nu \to \infty); \tag{216}$$

and even, for any fixed integer p,

$$\lambda_{\nu+1} - \sigma = o\big((\lambda_\nu - \sigma)(\lambda_{\nu-1} - \sigma) \ldots (\lambda_{\nu-p} - \sigma)\big) \qquad (\nu \to \infty). \tag{217}$$

Theorem 3. *Suppose that under the conditions of Theorem 2 all eigenvalues of* A *are real and correspond to linear elementary divisors. Suppose further that the following condition is satisfied:*

(α) σ *is not an arithmetic mean of two different eigenvalues of* A.

Then we have either for a certain constant K

$$\lambda_{\nu+1} - \sigma = O\Big(K^\nu \prod_{\varkappa=0}^{\nu} (\lambda_\varkappa - \sigma)^2\Big) \qquad (\nu \to \infty) \tag{218}$$

or

$$\frac{\lambda_{\nu+1} - \sigma}{(\lambda_\nu - \sigma)^2} \to g \neq 0 \qquad (\nu \to \infty). \tag{218°}$$

If on the other hand the condition (α) *is not satisfied, either* (218) *holds or the relation*

$$\frac{\lambda_{\nu+1} - \sigma}{(\lambda_{\nu-1} - \sigma)^2 (\lambda_\nu - \sigma)} \to g \neq 0 \qquad (\nu \to \infty) \tag{218°°}$$

either for even ν *or for odd* ν.

60. Since, as we assumed, to σ correspond only linear elementary divisors of A, there exists a non-singular matrix S such that we have

$$S A S^{-1} = \sigma I_m \dotplus (V + \sigma I_{n-m}), \tag{219}$$

where V is an $(n-m) \times (n-m)$ matrix in the Jordan canonical form, all eigenvalues of which are $\neq 0$.

Put

$$\delta_\nu = \sigma - \lambda_\nu \qquad (\nu = 0, 1, \ldots) \tag{220}$$

and

$$P_\nu = \prod_{\varkappa = 0}^{\nu} \delta_\varkappa \quad (\nu = 0, 1, \ldots), \qquad P_{-1} = 1. \tag{221}$$

If we multiply (212) from the left by $S P_{\nu-1}$, we can rewrite it, using (219) and (221), in the form

$$[\delta_{\nu-1} I_m \dotplus (V + \delta_{\nu-1} I_{n-m})] P_{\nu-1} S \xi_\nu = \delta_{\nu-1} P_{\nu-2} S \xi_{\nu-1};$$

or, replacing ν by $\nu+1$ and putting

$$P_{\nu-1} S \xi_\nu = \zeta_\nu \qquad (\nu = 0, 1, \ldots), \tag{222}$$

$$[\delta_\nu I_m \dotplus (V + \delta_\nu I_{n-m})] \zeta_{\nu+1} = \delta_\nu \zeta_\nu \qquad (\nu = 0, 1, \ldots). \tag{223}$$

Express ζ_ν according to the decomposition (219) as a direct sum

$$\zeta_\nu = \beta_\nu \dotplus \alpha_\nu; \tag{224}$$

where β_ν is an m-dimensional and α_ν an $(n-m)$-dimensional vector.

Introducing this into (223), we obtain two equations

$$\delta_\nu I_m \beta_{\nu+1} = \delta_\nu \beta_\nu, \tag{225}$$

and

$$(V + \delta_\nu I_{n-m}) \alpha_{\nu+1} = \delta_\nu \alpha_\nu. \tag{226}$$

But (225) gives $\beta_{\nu+1} = \beta_\nu$ and therefore we have $\beta_\nu = \beta$ $(\nu = 0, 1, \ldots)$, putting $\beta_0 = \beta$. By (222), the decomposition (224) now becomes

$$\zeta_\nu = P_{\nu-1} S \xi_\nu = \beta \dotplus \alpha_\nu. \tag{227}$$

61. From (227) we have

$$\overline{P}_{\nu-1} P_{\nu-1} \xi_\nu^* \xi_\nu = (\beta^* \dotplus \alpha_\nu^*) S^{*-1} S^{-1} (\beta \dotplus \alpha_\nu)$$

or, putting

$$D = S^{*-1} S^{-1}, \tag{228}$$

where D is a *positive* Hermitian matrix,

$$|P_{\nu-1}|^2 \xi_\nu^* \xi_\nu = \zeta_\nu^* D \zeta_\nu. \tag{229}$$

Further, by (227) and (219) we have

$$|P_{\nu-1}|^2 \xi_\nu^* A \xi_\nu - \sigma |P_{\nu-1}|^2 \xi_\nu^* \xi_\nu = \overline{P}_{\nu-1} P_{\nu-1} \xi_\nu^* (A - \sigma I_n) \xi_n$$
$$= \zeta_\nu^* S^{*-1} S^{-1} S (A - \sigma I_n) S^{-1} \zeta_\nu = \zeta_\nu^* D (O_m \dotplus V) \zeta_\nu,$$

where O_m is the $(m \times m)$-matrix consisting of zeros. We have finally

$$|P_{\nu-1}|^2 (\xi_\nu^* A \xi_\nu - \sigma \xi_\nu^* \xi_\nu) = \zeta_\nu^* D (O_m \dotplus V) \zeta_\nu. \tag{230}$$

If we subtract σ from both sides of (213) and multiply the numerator and the denominator in the difference on the right by $|P_{\nu-1}|^2$, we obtain the expression (230) as the numerator and the expression (229) as the denominator, so that

$$\lambda_\nu - \sigma = \frac{\zeta_\nu^* D (O_m \dotplus V) \zeta_\nu}{\zeta_\nu^* D \zeta_\nu}. \tag{231}$$

62. We now write D in block form corresponding to the decomposition (219),

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{232}$$

and then obtain by (227) for the denominator in (231)

$$\begin{aligned} \zeta_\nu^* D \zeta_\nu &= (\beta^* \dotplus \alpha_\nu^*) \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} (\beta \dotplus \alpha_\nu) \\ &= (\beta^* D_{11} + \alpha_\nu^* D_{21} \dotplus \beta^* D_{12} + \alpha_\nu^* D_{22}) (\beta \dotplus \alpha_\nu) \\ &= \beta^* D_{11} \beta + \alpha_\nu^* D_{21} \beta + \beta^* D_{12} \alpha_\nu + \alpha_\nu^* D_{22} \alpha_\nu . \end{aligned}$$

Putting

$$|\beta|^2 D_0 = \beta^* D_{11} \beta, \quad |\beta|^2 D_1 = \alpha_\nu^* D_{21} \beta + \beta^* D_{12} \alpha_\nu, \quad |\beta|^2 D_2 = \alpha_\nu^* D_{22} \alpha_\nu, \tag{233}$$

we see that the denominator in (231) is

$$|\beta|^2 (D_0 + D_1 + D_2).$$

As to the numerator in (231), by (232) we obtain

$$D(O_m \dotplus V) = \begin{pmatrix} O_m & D_{12} V \\ O & D_{22} V \end{pmatrix}$$

and

$$(\beta^* \dotplus \alpha_\nu^*) \begin{pmatrix} O & D_{12} V \\ O & D_{22} V \end{pmatrix} (\beta \dotplus \alpha_\nu) = \beta^* D_{12} V \alpha_\nu + \alpha_\nu^* D_{22} V \alpha_\nu .$$

Therefore, if we put

$$|\beta|^2 N_1 = \beta^* D_{12} V \alpha_\nu, \qquad |\beta|^2 N_2 = \alpha_\nu^* D_{22} V \alpha_\nu, \tag{234}$$

we have

$$\lambda_\nu - \sigma = \frac{N_1 + N_2}{D_0 + D_1 + D_2}. \tag{235}$$

On the other hand, putting

$$V^{-1} = W, \tag{236}$$

we have from (226)

$$V(I_{n-m} + \delta_\nu W) \alpha_{\nu+1} = \delta_\nu \alpha_\nu ,$$

and replacing ν by $\nu - 1$,

$$\alpha_\nu = \delta_{\nu-1} W (I + \delta_{\nu-1} W)^{-1} \alpha_{\nu-1}. \tag{237}$$

63. From (227) we have

$$\zeta_0 = S \xi_0 = \beta \dotplus \alpha_0 .$$

Denote the square roots of the smallest and greatest eigenvalues of the positive Hermitian matrix $S^* S$ by Λ_0 and Λ_1. Then putting $S\eta = \tau$, from (215) we have

$$\Lambda_1 |\xi_0 - \eta| \geqq |\zeta_0 - \tau|, \qquad |\tau| \geqq \Lambda_0 |\eta|,$$

$$|\zeta_0 - \tau| \leqq \frac{\Lambda_1}{\Lambda_0} \delta^{(0)} |\tau|. \tag{238}$$

Putting $n-m=l$, let the vectors α_0' and β' be in their corresponding spaces

$$\beta' = (x_1', \ldots, x_m'), \qquad \alpha_0' = (y_1, \ldots, y_l);$$

and for the n-dimensional eigenvector τ of (238),

$$\tau' = (x_1, \ldots, x_m, 0, \ldots, 0).$$

Writing

$$\varepsilon = \frac{\Lambda_1}{\Lambda_0}\delta^{(0)}, \qquad |\zeta_0| = p, \tag{239}$$

we have then by (238)

$$\sum_{\lambda=1}^{l} |y_\lambda|^2 + \sum_{\varkappa=1}^{m} |x_\varkappa' - x_\varkappa|^2 \leqq \varepsilon^2 |\tau|^2,$$

$$\sum_{\lambda=1}^{l} |y_\lambda|^2 + \sum_{\varkappa=1}^{m} |x_\varkappa'|^2 + \sum_{\varkappa=1}^{m} |x_\varkappa|^2 \leqq \sum_{\varkappa=1}^{m} (x_\varkappa \bar{x}_\varkappa' + x_\varkappa' \bar{x}_\varkappa) + \varepsilon^2 |\tau|^2,$$

$$p^2 + (1-\varepsilon^2)|\tau|^2 \leqq \left| \sum_{\varkappa=1}^{m} (x_\varkappa \bar{x}_\varkappa' + x_\varkappa' \bar{x}_\varkappa) \right|;$$

further, if we apply the Cauchy-Schwarz inequality and assume $\varepsilon < 1$,

$$p^2 + (1-\varepsilon^2)|\tau|^2 \leqq 2|\beta|\,|\tau|,$$

$$\frac{|\beta|}{p\sqrt{1-\varepsilon^2}} \geqq \frac{1}{2}\left(\frac{p}{\sqrt{1-\varepsilon^2}\,|\tau|} + \frac{\sqrt{1-\varepsilon^2}\,|\tau|}{p}\right) \geqq 1,$$

$$|\beta|^2 \geqq (1-\varepsilon^2)\,p^2 = (1-\varepsilon^2)\left(|\alpha_0|^2 + |\beta|^2\right),$$

$$|\alpha_0|^2 \leqq \frac{\varepsilon^2}{1-\varepsilon^2}|\beta|^2. \tag{240}$$

64. Now we put

$$\gamma_\nu = \frac{|\alpha_\nu|}{|\beta|}. \tag{241}$$

Since D in (228) is positive and therefore also D_{11} and D_{22} in (232) are positive, from (233) and (234) we have

$$D_0 \geqq p_0, \qquad D_1 \leqq p_1\gamma_\nu, \qquad D_2 > 0,$$

$$|N_1| \leqq p_2\gamma_\nu, \qquad |N_2| \leqq p_3\gamma_\nu^2,$$

where p_0, p_1, p_2, p_3 are positive constants depending only on A. Thence by (235)

$$|\lambda_\nu - \sigma| = |\delta_\nu| \leqq \frac{p_2 + p_3\gamma_\nu}{p_0 - p_1\gamma_\nu}\gamma_\nu, \tag{242}$$

provided $\gamma_\nu < \dfrac{p_0}{p_1}$.

Put

$$c_1 = \operatorname{Max}\left(1, \frac{2}{p_0}(p_2 + p_3)\right). \tag{243}$$

Denote by c the exact bound, $\|W\|$, of the matrix W in (237), *i.e.*, the square root of the maximal eigenvalue of WW^*, and put

$$c_2 = \operatorname{Max}(1, 2c). \tag{244}$$

Finally, put

(245) $$c_3 = \operatorname{Min}\left(\frac{1}{2}, \frac{1}{2}\frac{p_0}{p_1}, \frac{1}{c_1 c_2}\right).$$

We denote by $\varkappa$ a fixed number such that $0 < \varkappa < 1$ and assume that for a $\nu \geqq 1$ we have

(246) $$|\delta_{\nu-1}| \leqq \frac{\varkappa}{c_2} < \frac{1}{2c}.$$

Then it follows that

$$\|W(I + \delta_{\nu-1} W)^{-1}\| \leqq \sum_{\mu=0}^{\infty} |\delta_{\nu-1}|^{\mu} \|W\|^{\mu+1} \leqq 2c,$$

and therefore from (237) and (241)

$$|\alpha_\nu| \leqq c_2 |\delta_{\nu-1}| \, |\alpha_{\nu-1}|,$$

(247) $$\gamma_\nu \leqq c_2 |\delta_{\nu-1}| \gamma_{\nu-1} \leqq \varkappa \gamma_{\nu-1}.$$

Further, if we assume for a $\nu \geqq 1$ that

(248) $$\gamma_\nu \leqq c_3,$$

it follows from (242) that

(249) $$|\delta_\nu| \leqq \frac{2}{p_0}(p_2 + p_3)\gamma_\nu \leqq c_1 \gamma_\nu.$$

65. We now make the assumptions

(250) $$\gamma_0 \leqq c_3,$$

(251) $$|\delta_0| \leqq \frac{\varkappa}{c_2};$$

hence we shall prove that

(252) $$\gamma_\nu \leqq c_3 \varkappa^{2^{\nu-1}} \qquad (\nu = 1, 2, \ldots),$$

(253) $$|\delta_\nu| \leqq \frac{1}{c_2} \varkappa^{2^{\nu-1}} \qquad (\nu = 1, 2, \ldots).$$

When $\nu = 1$, (252) follows from (247), since

$$\gamma_1 \leqq \varkappa \gamma_0 \leqq c_3 \varkappa.$$

If we now use (249) when $\nu = 1$, we have from (248)

$$|\delta_1| \leqq c_1 \gamma_1 \leqq c_1 c_3 \varkappa \leqq \frac{\varkappa}{c_2},$$

since by (245) $c_1 c_3 \leqq \frac{1}{c_2}$. Assume now that we have

(252°) $$\gamma_{\nu-1} \leqq c_3 \varkappa^{2^{\nu-2}},$$

and

(253°) $$|\delta_{\nu-1}| \leqq \frac{1}{c_2} \varkappa^{2^{\nu-2}}.$$

Then, since (246) is satisfied, we have

$$\gamma_\nu \leqq c_2 \frac{\varkappa^{2^{\nu-2}}}{c_2} c_3 \varkappa^{2^{\nu-2}} = c_3 \varkappa^{2^{\nu-1}}$$

from (247), and (252) is proved. Now since (248) holds, we can use (249); combining it with (252), we obtain

$$|\delta_\nu| \leq c_1 \gamma_\nu \leq c_1 c_3 \varkappa^{2^{\nu-1}},$$

and (253) is proved.

We wish now to choose $\delta^{(0)}$ in such a way that the conditions (250), (251) are satisfied. But from (240) and (239), using (241) for $\nu=0$, we have

$$\gamma_0 \leq \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \leqq \frac{\frac{\varLambda_1}{\varLambda_0}\delta^{(0)}}{\sqrt{1-\frac{\varLambda_1^2}{\varLambda_0^2}\delta^{(0)\,2}}}.$$

The right-hand bound is certainly less than c_3 if

$$\frac{\varLambda_1}{\varLambda_0}\delta^{(0)} \leqq \frac{2}{3} c_3, \qquad \delta^{(0)} \leqq \frac{2}{3} c_3 \frac{\varLambda_0}{\varLambda_1},$$

since by (245) $c_3 \leqq \frac{1}{2}$. On the other hand, (251) follows from (214) if $\delta^{(0)} \leqq \frac{\varkappa}{c_2}$ with $0<\varkappa<1$. Therefore it is sufficient to choose

$$\delta^{(0)} = \operatorname{Min}\left(\frac{\varkappa}{c_2}, \frac{2}{3} c_3 \frac{\varLambda_0}{\varLambda_1}\right)$$

to insure the formulae (252) and (253). Theorem 1 is now proved.

66. To prove Theorem 2, assume first that the inequalities (250) and (251) hold. Then we have from (249) and (247)

$$|\delta_\nu| \leqq c_1 \gamma_\nu \leqq c_1 c_2 \gamma_{\nu-1} |\delta_{\nu-1}|,$$

and this proves (216). Using (247) p times, we obtain

$$|\delta_\nu| \leqq c_1 c_2 |\delta_{\nu-1}| \gamma_{\nu-1} \leqq c_1 |\delta_{\nu-1} \delta_{\nu-2}| c_2^2 \gamma_{\nu-2} \leqq \cdots \leqq c_1 |\delta_{\nu-1} \delta_{\nu-2} \ldots \delta_{\nu-p}| c_p^p \gamma_{\nu-p},$$

and this proves (217) since $\gamma_{\nu-p} \to 0$.

If the initial conditions (250), (251) are not satisfied but $\delta_\nu \to 0$, it is sufficient to prove that for a certain index k we have

$$|\delta_k| \leqq \frac{\varkappa}{c_2}, \qquad |\gamma_k| \leqq c_3.$$

Since $\delta_\nu \to 0$, the inequality (246) with $\varkappa < \frac{1}{2}$ certainly holds from a certain ν onward, and since from there on (247) is valid, $\gamma_\nu \to 0$. Theorem 2 is completely proved.

67. In proving Theorem 3, by what has been said in the proof of Theorem 2, we can assume without loss of generality that the conditions (250), (251) are satisfied and therefore also the relations (247) and (249), since otherwise we could replace ν by $\nu+k$.

By (241) and (233) we obviously have

$$D_1 = O(\gamma_\nu), \qquad D_2 = O(\gamma_\nu^2),$$

and therefore in virtue of (232) $D_0+D_1+D_2 \to D_0$. Thus by (235) since $D_0>0$,

(254) $$D_0(\lambda_\nu - \sigma) = (1+\varepsilon_\nu)(N_1+N_2), \qquad \varepsilon_\nu \to 0.$$

From (234), $N_2 = O(\gamma_\nu^2)$, and since by (247)

(255) $$\gamma_\nu \leqq c_2^\nu |P_{\nu-1}| \gamma_0,$$

finally from (249)

(256) $$\delta_\nu = O(c_2^{2\nu} P_{\nu-1}),$$

(257) $$N_2 = O(c_2^{2\nu} P_{\nu-1}^2).$$

We wish now to discuss N_1, given by (234). Introducing the vector γ by

(258) $$\gamma^* = \frac{1}{|\beta|^2} \beta^* D_{12} V,$$

we have for the $(n-m)$-dimensional vectors γ^* and α_ν:

(259) $$N_1 = \gamma^* \alpha_\nu.$$

If the vector γ defined by (258) vanishes, then by (255) we have from (254)

$$\lambda_\nu - \sigma = O(N_2) = O(c_2^{2\nu} P_{\nu-1}^2),$$

and we have the assertion (218) of Theorem 3. Discarding this case, we can assume $\gamma^* \neq 0$.

68. Under the assumptions of Theorem 3, the matrix V in (219) is a diagonal matrix with the elements $\sigma_\mu - \sigma$ along the diagonal, $\mu = 1, 2, \ldots, n-m$, where $\sigma_1, \ldots, \sigma_{n-m}$ are the fundamental roots of A, different from σ. Denote the μ^{th} components of γ^* and α_ν respectively by $\gamma^{*(\mu)}$, $\alpha_\nu^{(\mu)}$. Then, writing $\nu - 1$ instead of ν, from (226) we have

$$\alpha_\nu^{(\mu)} = \frac{\alpha_{\nu-1}^{(\mu)} \delta_{\nu-1}}{\sigma_\mu - \lambda_{\nu-1}} \qquad (\nu = 1, \ldots; \ \mu = 1, \ldots, n-m),$$

$$\alpha_\nu^{(\mu)} = \alpha_0^{(\mu)} P_{\nu-1} \prod_{\varkappa=0}^{\nu-1} (\sigma_\mu - \lambda_\varkappa)^{-1}.$$

Therefore, putting

(260) $$P_\nu^{(\mu)} = \prod_{\varkappa=0}^{\nu} (\sigma_\mu - \lambda_\varkappa),$$

by (259) we have

(261) $$N_1 = \sum_{\mu=1}^{n-m} \frac{\gamma^{*(\mu)} \alpha_0^{(\mu)}}{P_{\nu-1}^{(\mu)}} P_{\nu-1}.$$

From (260)

(262) $$P_{\nu-1}^{(\mu)} (\sigma_\mu - \sigma)^{-\nu} = \prod_{\varkappa=0}^{\nu-1} \left(1 + \frac{\delta_\varkappa}{\sigma_\mu - \sigma}\right) = T_\nu^{(\mu)}.$$

Since, by (253), the series $\sum\limits_{\varkappa=0}^{\infty} |\delta_\varkappa|$ is obviously convergent, the products

$$\Gamma_\mu = \prod_{\varkappa=0}^{\infty} \left(1 + \frac{\delta_\varkappa}{\sigma_\mu - \sigma}\right)$$

are also convergent and have values different from zero, since none of the $\sigma_\mu - \lambda_\varkappa$ vanishes. Further, putting

(263) $$\Gamma_\mu^{(\nu)} = \frac{\Gamma_\mu}{T_\nu^{(\mu)}} = \prod_{\varkappa=\nu}^{\infty} \left(1 + \frac{\delta_\varkappa}{\sigma_\mu - \sigma}\right),$$

we have
$$\Gamma_\mu^{(\nu)} - \Gamma_\mu^{(\nu+1)} = \frac{\delta_\nu}{\sigma_\mu - \sigma}\, \Gamma_\mu^{(\nu+1)}.$$

Therefore the series $\sum\limits_{\varkappa=\nu}^{\infty} (\Gamma_\mu^{(\varkappa)} - \Gamma_\mu^{(\varkappa+1)})$ is convergent and $O(\delta_\nu)$. But this series is $\Gamma_\mu^{(\nu)} - 1$, and therefore

$$\Gamma_\mu^{(\nu)} = 1 + O(\delta_\nu). \tag{264}$$

69. It follows now from (263), (264) and (262) that
$$T_\nu^{(\mu)} = \Gamma_\mu (1 + O(\delta_\nu)),$$
$$P_{\nu-1}^{(\mu)} (\sigma_\mu - \sigma)^{-\nu} = \Gamma_\mu (1 + O(\delta_\nu)),$$
and we have

$$N_1 = \sum_{\mu=1}^{n-m} P_{\nu-1} \frac{d_\mu}{(\sigma_\mu - \sigma)^\nu} (1 + O(\delta_\nu)) \tag{265}$$

by putting
$$d_\mu = \frac{\gamma^{*(\mu)} \alpha_0^{(\mu)}}{\Gamma_\mu} \qquad (\mu = 1, \ldots, n-m).$$

We now put
$$K = \mathrm{Max}\left(\frac{1}{|\sigma_\mu - \sigma|} \quad (\mu = 1, \ldots, n-m),\ c_2,\ c_2^2\right),$$
keep in the sum (265) only the terms for which $d_\mu \neq 0$, and combine the terms with same value of $\sigma_\mu - \sigma$. These terms will be now written changing the order of the σ_μ, with the index τ going from 1 to t, and we obtain thus

$$\frac{N_1^{(\nu)}}{P_{\nu-1}} = \sum_{\tau=1}^{t} \frac{d_\tau}{(\sigma_\tau - \sigma)^\nu} + O(\delta_\nu K^\nu), \tag{266}$$

where now we have added the superscript ν to N_1 obtained from (234) and (261). The σ_τ in (266) are now all different one from another.

We can also assume that

$$|\sigma_1 - \sigma| \leqq |\sigma_2 - \sigma| \leqq \cdots \leqq |\sigma_t - \sigma|. \tag{267}$$

If in (266) we have $t = 0$, then by (254), (256) and (257) we get
$$\delta_\nu = O(N_1 + N_2) = O(P_{\nu-1} K^{2\nu} \delta) + O(K^{2\nu} P_{\nu-1}^2),$$
$$\delta_\nu (1 + O(P_{\nu-1} K^{2\nu})) = O(K^{2\nu} P_{\nu-1}^2),$$
and since $P_{\nu-1} K^{2\nu} \to 0$ by (253),
$$\delta_\nu = O(K^2 P_{\nu-1}^2),$$

i.e., (218). Discarding this case, we can now assume $t \geqq 1$.

70. Assume now that the condition (α) of Theorem 3 is satisfied. Then the equality signs in (267) are excluded, and from (266) follows
$$N_1^{(\nu)} \sim P_{\nu-1} \frac{d_1}{(\sigma_1 - \sigma)^\nu},$$
and by (254) and (257)

$$\lambda_\nu - \sigma \sim \frac{d_1}{D_0} \frac{1}{(\sigma_1 - \sigma)^\nu} P_{\nu-1}, \tag{268}$$

and therefore
$$\frac{\sigma-\lambda_{\varkappa+1}}{\sigma-\lambda_{\varkappa}} \sim \frac{P_{\varkappa}}{P_{\varkappa-1}}\,\frac{1}{\sigma_1-\sigma} = \frac{\sigma-\lambda_{\varkappa}}{\sigma_1-\sigma} \qquad (\varkappa\to\infty);$$
the assertion (218°) follows now with $g=\frac{1}{\sigma-\sigma_1}$.

71. If, finally, the condition (α) is not satisfied, but in (267) $|\sigma_1-\sigma|<|\sigma_2-\sigma|$, then (218°) still holds, and therefore (218°°) holds both for even ν and for odd. If on the other hand
$$|\sigma_1-\sigma|=|\sigma_2-\sigma|, \qquad \sigma_1-\sigma=-(\sigma_2-\sigma),$$
(266) becomes
$$\frac{N_1^{(\nu)}}{P_{\nu-1}} = \frac{d_1+(-1)^\nu d_2}{(\sigma_1-\sigma)^\nu} + o\left(\frac{1}{(\sigma_1-\sigma)^\nu}\right).$$
But then, since $d_1\neq 0$ and $d_2\neq 0$, either $d_1+d_2\neq 0$ or $d_1-d_2\neq 0$. In the first case *for even* ν we have
$$\frac{N_1^{(\nu)}}{P_{\nu-1}} \sim \frac{d_1+d_2}{(\sigma_1-\sigma)^\nu},$$
and (218°°) follows immediately. In the second case *for odd* ν we obtain
$$\frac{N_1^{(\nu)}}{P_{\nu-1}} \sim \frac{d_1-d_2}{(\sigma_1-\sigma)^\nu}$$
and (218°°) follows again. Theorem 3 is now completely proved.

72. We apply the method discussed in this part to the matrix (177) treated in § 46 by the generalized Rayleigh quotient iteration. Here, using only the sequence of vectors ξ_k and introducing again μ_k and σ_k by (178) and (179), we obtain the formulae

$$\mu_\nu = \frac{5-3\sigma_\nu}{2-2\sigma_\nu+\sigma_\nu^2}, \qquad \sigma_{\nu+1} = \frac{\mu_\nu\,\sigma_\nu}{1+\mu_\nu+2\sigma_\nu}. \tag{269}$$

Thus, starting with $\sigma_0=1$, we get the following values:

$$\begin{array}{ll}
\sigma_0 = 1 & \mu_0 = 2 \\
\sigma_1 = \frac{2}{5} & \mu_1 = 1.117647 \\
\sigma_2 = 0.153228 & \mu_2 = .405180 \\
\sigma_3 = .0362723 & \mu_3 = .0919831 \\
\sigma_4 = .002865057 & \mu_4 = .007170851 \\
\sigma_5 = .0^4 20283226 & \mu_5 = .0^4 50708.
\end{array}$$

For the values of $\frac{\mu_\nu}{\mu_{\nu-1}^2}$, $\nu=1, \ldots, 5$, we have .28, .324, .56, .848, .986.

The convergence to 1 is apparent although not very fast. In order to improve the last result, μ_5, we apply to μ_3, μ_4, μ_5 the argument of § 19 of Part II, replacing the exponents 4 and 3 by 3 and 2 in formula (61). Then we obtain the improved value of μ_5, $\mu_5' = .0^5 7998$.

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Erratum

Berichtigung zu W. Gerisch, Zum Problem der an den Rändern fest eingespannten, elastischen Platte, Vol. 2, 227—242 (1958/59).

Auf S. 240 sind die Fig. 2 und 3 vertauscht.

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